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September 2016

Abstract

This paper theoretically investigates which auctions competing sellers select when they can choose between first-price and second-price auctions, and when risk averse bidders endogenously enter one of the auctions. We first consider bidders' entry decisions between exogenously given auctions, and find that there exists a symmetric entry equilibrium that is characterized by a mixed strategy, which depends on the bidders' degree of absolute risk aversion. In a next step, we endogenize the sellers' choice of auctions. We show that competing sellers have a dominant strategy to select first-price auctions if bidders exhibit nondecreasing absolute risk aversion. If bidders exhibit decreasing absolute risk aversion, however, other equilibria exist in which sellers select second-price auctions as well.

Keywords: Auctions, Entry decisions, Risk aversion

JEL classification: D44, D81

Acknowledgements

We are grateful to Stephanie Rosenkranz, Britta Hoyer and Swantje Falcke for useful comments and suggestions. We would also like to thank the participants of the TKI Seminar Series in Utrecht, the Maastricht Lecture Series in Economics, the 2014 International Workshop on Game Theory and Economic Applications of the Game Theory Society in São Paulo, the 2015 Conference on Economic Design in Istanbul, and the 2015 International Conference on Game Theory in Stony Brook. All remaining errors are ours.

1. Introduction

The use of auctions as a means of selling goods has traditionally been confined to specific goods such as art objects and agricultural products. The rise of the Internet, however, has led to a reduction in transaction costs and better matching of supply and demand, and has thereby created new markets for auctions (Ockenfels et al., 2006). Nowadays, a vast volume of economic transactions is conducted through online auctions. Online marketplaces, like eBay and Yahoo, offer a multitude of simultaneous auctions in which goods such as collectibles and phones are sold; specialized online auction stores sell goods for which well-established markets already exist such as holidays, concert tickets and computers. The development that units of a homogeneous good are now sold through multiple auction formats allows consumers to choose which mechanism to enter. As a result, auctioneers find themselves competing against one another to attract consumers.

The economics literature has long treated auctions as isolated, studying how a single seller facing multiple bidders can maximize his revenue or, in the case of procurement auctions, how a single bidder facing multiple sellers can maximize her utility. Auction theory's most celebrated results, the revenue ranking theorems, compare the expected revenues of different auction formats while treating the number of bidders in each auction as given (e.g. Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981; Maskin and Riley, 1984). Though these results have proven to be very valuable for the design of auctions for isolated sales such as the spectrum auctions, the traditional revenue ranking theorems may no longer apply if auctioneers operate in a competitive market, where the ability to attract bidders is a crucial determinant of an auction's success (e.g. Klemperer, 2002; Ivanova-Stenzel and Salmon, 2008a). After all, an auction that in isolation generates the highest revenue may no longer do so if bidders have no incentive to enter this auction. "In practice, auctions [...] often fail because of insufficient interest by bidders" (Milgrom, 2004, p.209). An auctioneer operating in a competitive market should therefore consider bidders' preferences, as well as the selling mechanisms his competitors offer, when deciding which auction to offer.

The aim of this paper is to study the auction selection problem of competing auctioneers. That is, we theoretically investigate which auctions are selected by auctioneers when they operate in a competitive market and when bidders endogenously enter auctions. In doing so, we consider an auction selection game consisting of three stages. At Stage 1 of the game, sellers decide which auctions to offer. At Stage 2, the bidders learn which auctions are offered and enter one of them. At Stage 3, the auctions are conducted.

Throughout this paper, we make the following modeling assumptions on sellers. We consider risk neutral sellers who simultaneously offer a single unit of a homogeneous good in sealed bid auctions. More precisely, sellers may choose to offer a first-price auction or a second-price auction. These auctions and their dynamic counterparts (the Dutch and English auction, respectively) are frequently used both on and off the Internet and have, for that reason, also attracted considerable attention in the theoretical and experimental literature. In the main analysis we restrict the number of sellers to two, but we later show that qualitatively similar results can be obtained when there are more than two sellers.

On the bidders' side, we assume that bidders demand one unit of the good and choose to enter one of the auctions. They cannot choose to opt out of the auction or enter both auctions instead. Additionally, the bidders are ex ante symmetrically informed. This means that before entering an auction bidders do not know their own value for the good, which is both independent and private, but they do know the distribution of values.¹ Furthermore, bidders know whether and to which extent they are risk averse, but in the main analysis we assume that bidders are homogeneous in this respect. This implies that bidders cannot make their entry decisions dependent on any private information they may have. In an extension, however, we discuss the implications of allowing bidders to be heterogeneously risk averse.

¹This is a common assumption in much of the theoretical and experimental literature studying entry into auctions (e.g. McAfee and McMillan, 1987b; Engelbrecht-Wiggans, 1987, 1993; Levin and Smith, 1994; Smith and Levin, 1996; Pevnitskaya, 2004; Palfrey and Pevnitskaya, 2008; Ivanova-Stenzel and Salmon, 2004a,b, 2008a,b). The assumption is motivated by examples where bidders may only learn their exact value for the good for sale once they actually participate in the auction. Pevnitskaya (2004) gives an example of antique auctions, where sellers often advertise general inventory and where bidders can determine their exact value only after coming to the auction house and examining the goods prior to sale. As a result, these bidders do know the distribution of values, but only know their independent private value after entering the auction.

Various studies have analyzed the role of risk aversion in auctions and have shown that it is a critical factor explaining why bidders may not be indifferent between various auction formats. Theoretical research predicts that risk aversion results in overbidding in first-price auctions but does not change the equilibrium bidding strategy in second-price auctions (e.g. Riley and Samuelson, 1981; Maskin and Riley, 1984; Cox et al., 1985, 1988). As such, it also affects the utility bidders can expect from participating in these auctions (Matthews, 1983, 1987). Previous experimental studies have shown that risk aversion may play a role in bidders' entry decisions between different auctions, although the results are contingent on the experimental design.² Our study aims to provide the theoretical foundations for these findings and additionally explores the implications for the auction selection problem of competing auctioneers.

The auction selection game is solved using backward induction. We use existing results on bidding strategies and bidder preferences among auctions to analyze bidders' entry decisions in Stage 2. In doing so, we extend the models of endogenous entry of Levin and Smith (1994), Smith and Levin (1996) and Pevnitskaya (2004), who model entry as a symmetric equilibrium involving mixed strategies. We find that the probability of entering each auction depends on the bidders' degree of absolute risk aversion. More specifically, when bidders decide between entering a first-price and a second-price auction, each auction is entered with equal probability if bidders are risk neutral or exhibit constant absolute risk aversion. However, if bidders exhibit decreasing absolute risk aversion, they are more likely to enter the second-price auction; if bidders exhibit increasing absolute risk aversion, they are more likely to enter the first-price auction. As risk averse bidders overbid in first-price auctions but not in second-price auctions, these findings imply that in Stage 1 both sellers prefer to offer first-price auctions when bidders exhibit nondecreasing absolute risk aversion. However, when bidders exhibit decreasing risk aversion, other auction selection equilibria may exist.

Our study adds to the literature on bidder preferences and endogenous entry, as well as to the literature on competing auctions. Whereas auction theorists have traditionally focused on the seller's perspective, researchers are now also taking the bidder's point of view. It can be seen that Myerson's (1981) proof for the revenue equivalence between first-price and second-price auctions follows from a utility equivalence for risk neutral bidders. Risk neutral bidders are thus indifferent between first-price and second-price auctions. Matthews (1983, 1987) compares the utility of bidders with different degrees of absolute risk aversion. He finds that bidders who exhibit constant absolute risk aversion are also indifferent between first-price and second-price auctions. This result is later generalized by Monderer and Tennenholtz (2004) for all k -price auctions and by Hon-Snir (2005) for all standard auctions.³ Hon-Snir additionally shows that the utility equivalence for risk averse bidders holds if and only if bidders exhibit constant absolute risk aversion. This is consistent with the findings of Matthews (1987), who shows that bidders with decreasing absolute risk aversion prefer second-price auctions and bidders with increasing absolute risk aversion prefer first-price auctions.

The theoretical literature on entry into auctions studies the decision whether or not to enter an auction with an entry fee or when there exists an outside option. The literature can roughly be divided into two strands. The first strand assumes that bidders do not possess any private information before deciding to enter an auction or not. In this case, the theoretical literature focuses on two types of equilibria. McAfee and McMillan (1987b) and Engelbrecht-Wiggans (1987, 1993) focus on deterministic, asymmetric equilibria involving pure entry strategies. This approach results in a plethora of equilibria, where a subset of bidders enters the auction and another subset does not. The process by which symmetric bidders are divided into these subsets, however, is not identified. Levin and Smith (1994) and Smith and Levin (1996) therefore focus on a unique, stochastic, symmetric equilibrium involving mixed entry strategies. Various experimental and empirical studies have compared these two approaches and find that entry is best explained by the stochastic model (e.g. Smith and Levin, 2002; Bajari and Hortacsu, 2003; Reiley, 2005). The second strand

²Ivanova-Stenzel and Salmon (2004a, 2008b) find that risk aversion explains entry decisions between English and first-price auctions when each auction consists of only two bidders. In another set of experiments, the authors allow bidders to coordinate freely over the auctions (Ivanova-Stenzel and Salmon, 2008a, 2011). In these circumstances, risk aversion does not seem to explain entry decisions.

³A standard auction is defined as an auction in which the bidder with the highest bid wins.

of the literature assumes that bidders obtain some type of private information before making their entry decisions. This includes bidders' private values (Menezes and Monteiro, 2000) and bidders' heterogeneous degrees of risk aversion (Pevnitskaya, 2004; Palfrey and Pevnitskaya, 2008). These studies find that there is a unique entry equilibrium in pure strategies, which involves a cut-off value based on the bidders' private information. To the best of our knowledge, our study is the first to develop a theoretical model on entry decisions between different auction formats, although some experimental studies on this topic exist (e.g. Ivanova-Stenzel and Salmon, 2004a,b, 2008a,b, 2011; Engelbrecht-Wiggans and Katok, 2005).⁴

Most studies in the competing auctions literature analyze auction selection problems where the dimension along which sellers compete is the reserve price or the entry fee (e.g. McAfee, 1993; Peters and Severinov, 1997; Damianov, 2012). Instead, the dimension along which sellers compete in our study is the auction format itself. The study that is perhaps most closely related to ours is that of Monderer and Tenenholtz (2004), who theoretically investigate auction selection with bidders who exhibit constant absolute risk attitudes but assume exogenous random participation (McAfee and McMillan, 1987a). They find that sellers prefer to select a first-price auction when bidders exhibit constant absolute risk aversion. When bidders exhibit constant absolute risk seekingness, however, sellers will be better off selecting a k -price auction of higher order. Including a larger range of risk attitudes and assuming entry to be stochastic, allows us to obtain novel insights into auction selection and simultaneously add to existing revenue ranking results.

The remainder of this paper is structured as follows. Section 2 describes the model in detail. Section 3 analyzes the entry decisions in Stage 2 of our three-stage game and Section 4 analyzes the auction selection in Stage 1. Finally, Section 5 discusses some extensions of our model, and Section 6 discusses our findings and provides concluding remarks.

2. Model

Suppose that two sellers simultaneously offer a single unit of a homogeneous good to a group of $N \geq 2$ bidders. Each seller decides to offer the good either in a first-price auction (FPA) or in a second-price auction (SPA); bidders are free to enter either auction. We assume that sellers are risk neutral and have zero value for the good. Bidders are symmetric and homogeneous. More specifically, bidder i 's preferences are given by the utility function $u(m_i)$, which satisfies $u'(m_i) > 0$ and $u''(m_i) \leq 0$, and where m_i represents her payoff. Throughout this paper, we use r to refer to the Arrow-Pratt coefficient of absolute risk aversion, which is measured by $-\frac{u''(m_i)}{u'(m_i)}$.

We consider the following three-stage game, which is an extension of the models of endogenous entry by Levin and Smith (1994), Smith and Levin (1996), and Pevnitskaya (2004). At Stage 1, seller $l = \{1, 2\}$ selects auction $a_l = \{FPA, SPA\}$. At this stage, the number of bidders, N , their utility functions, $u(m_i)$, and the distribution of values, $F(v)$, are common knowledge. Prior to Stage 2, the N bidders learn a_l , i.e., they learn which auctions have been selected by the sellers. Subsequently, each of the N bidders enters one of the auctions: n_1 bidders enter a_1 and $n_2 = N - n_1$ bidders enter a_2 . At Stage 3, the auctions are conducted. Each bidder i learns n_l and receives her private value v_i , which is independently and identically distributed according to the common distribution function $F(v)$, with strictly positive density $f(v)$ on the interval $[v, \bar{v}]$. All bidders then simultaneously submit sealed bids according to the unique, symmetric and increasing Bayesian Nash equilibrium bidding function $b(v|a_l, n_l)$. The outcome of the auctions is to allocate the goods to the highest bidders. If bidder i wins the auction, she receives a payoff of $v_i - p_i$, where p_i represents i 's payment. Whereas in the FPA p_i is equal to i 's own bid, in the SPA it is equal to the bid of the second highest bidder. If bidder i does not win the auction, she receives a payoff of zero.

The outcomes of Stage 3 have been extensively analyzed in the literature (e.g. Vickrey, 1961; Riley and Samuelson, 1981; Maskin and Riley, 1984). In the FPA, the equilibrium bidding strategy when bidders are risk neutral is to bid an amount equal to the expectation of the highest of $n_l - 1$ values below one's own value. When bidders are risk averse, however, the equilibrium bidding strategy is higher. In the SPA, the

⁴Extensive overviews of the literature on entry into auctions can be found in Kagel and Levin (2014) and Aycinena et al. (2015).

equilibrium bidding strategy is to bid one's own value, regardless of whether bidders are risk averse or not. Applying backward induction, we use these outcomes to analyze the entry decisions in Stage 2 and the selection of auctions in Stage 1.

3. Endogenous entry

In this section, we analyze bidders' entry decisions in Stage 2 of the game. Let $E[u|a_l, n_l]$ denote each bidder's ex ante expected utility from entering auction a_l , learning n_l and, after learning her value v , bidding according to the symmetric equilibrium bidding strategy $b(v|a_l, n_l)$. Note that $E[u|a_l, n_l]$ is decreasing in n_l , because an increase in the number of bidders decreases the probability of winning and raises the payment in both FPA and SPA (Smith and Levin, 1996). Moreover, if a bidder is the only one in an auction, she will earn a positive payoff with certainty.

Following Levin and Smith (1994), Smith and Levin (1996) and Pevnitskaya (2004), we focus on a symmetric entry equilibrium. Any symmetric entry equilibrium will necessarily involve mixed strategies, such that each bidder enters a_1 with probability q^* and enters a_2 with probability $1 - q^*$.^{5,6} The reason for this is simple. Suppose that all bidders enter a_1 . Then each bidder has an incentive to switch to a_2 , as in this auction she will be the only bidder and thereby earn a positive payoff with certainty. The same holds if all bidders enter a_2 . In equilibrium, each bidder must therefore be indifferent between entering a_1 and a_2 . This implies that the symmetric entry equilibrium, given by $q^* \in (0, 1)$, is described by

$$\begin{aligned} & \sum_{n_1=1}^N \binom{N-1}{n_1-1} (q^*)^{n_1-1} (1-q^*)^{N-n_1} E[u|a_1, n_1] \\ &= \sum_{n_2=1}^N \binom{N-1}{n_2-1} (1-q^*)^{n_2-1} (q^*)^{N-n_2} E[u|a_2, n_2] \end{aligned} \quad (1)$$

where the left-hand side (LHS) of (1) gives the expected utility of entering a_1 and the right-hand side (RHS) gives the expected utility of entering a_2 . Furthermore, the terms in the brackets give the binomial probability that exactly $n_l - 1$ competing bidders also enter the auction, giving n_l in total. We find that the resulting equilibrium probability of entry is unique for a given r .

Lemma 1. *There exists a symmetric entry equilibrium in mixed strategies, such that each bidder enters auction a_1 with probability q^* and enters auction a_2 with probability $1 - q^*$. The equilibrium probability of entry is implicitly defined by (1) and is unique for a given risk parameter r .*

Proof. Define $z(q, r)$ as the function equal to the LHS minus the RHS of (1). From Lemma 1 of Pevnitskaya (2004, p.6), it immediately follows that the LHS of (1) is continuous and monotonically decreasing in q . The RHS of (1) is continuous and monotonically increasing in q (see Lemma A1 in Appendix A). This implies that $z(q, r)$ is continuous and monotonically decreasing in q .

Equilibrium is achieved when $z(q^*, r) = 0$. Notice that any q^* satisfying this condition must be in the interval $(0, 1)$. For instance, suppose that $q^* = 0$, such that all bidders enter a_2 . Then $z(q^*, r) > 0$ and each bidder can receive a positive payoff with certainty by entering a_1 . Conversely, suppose that $q^* = 1$, such

⁵Note that the actual number of bidders in a_1 then follows a binomial distribution with mean $q^*N = \bar{n}_1$ and variance $(1 - q^*)\bar{n}_1$. Similarly, the actual number of bidders in a_2 follows a binomial distribution with mean $(1 - q^*)N = \bar{n}_2$ and variance $q^*\bar{n}_2$.

⁶Even with symmetric bidders asymmetric entry equilibria may exist, where some subset of bidders enters a_1 with probability 1 and another subset enters a_{-l} with probability 1. Likewise, asymmetric equilibria may exist where some subset of bidders enters a_l with probability 1 and where another subset of bidders randomizes over the auctions with the symmetric entry probability q . However, note that it is not possible to have asymmetric equilibria where different bidders have different mixing probabilities, as (1) is identical for all bidders. Furthermore, note that the assumption of pure strategies may lead to very many equilibria, dependent on which subset of bidders enters a_l and which subset enters a_{-l} . This creates an equilibrium selection problem. Therefore, we solely focus on a symmetric entry equilibrium, which not only restores full symmetry to the model but also turns out to be unique.

that all bidders enter a_1 . Then $z(q^*, r) < 0$ and each bidder can receive a positive payoff with certainty by entering a_2 . As a result, only $0 < q^* < 1$ can satisfy $z(q^*, r) = 0$. By the intermediate value theorem it then follows that there exists a unique symmetric equilibrium probability of entry, q^* , and it is defined by (1). \square

The value of the equilibrium probability of entry, q^* , crucially depends on the type of auctions that are selected by the sellers and on the utility functions of the bidders. Lemma 2 and Proposition 1 give q^* for different circumstances.

Lemma 2. *Suppose that either $a_1 = a_2$, or $a_1 \neq a_2$ and bidders are risk neutral ($r = 0$). The symmetric entry equilibrium is then given by $q^* = 0.5$.*

Proof. Let $a_1 = a_2$. It follows immediately that $E[u|a_1, n_1] = E[u|a_2, n_2]$ for $n_1 = n_2$. Similarly, let $a_1 \neq a_2$ and $r = 0$. From the utility equivalence principle for risk neutral bidders, that follows from Myerson (1981), we know that $E[u|a_1, n_1] = E[u|a_2, n_2]$ for $n_1 = n_2$. As a result, each bidder's entry decision is only affected by the number of competing bidders in each auction. This leads bidders to randomize over auctions with equal probability, that is, $q^* = 0.5$. \square

When both sellers select FPAs or, equivalently, SPAs, then the ex ante expected utility of a_1 and a_2 is the same whenever the number of bidders in each auction is also the same. This implies that bidders are indifferent between entering a_1 and a_2 as long as $n_1 = n_2$. In equilibrium, bidders will therefore enter each auction with equal probability. When sellers select different auctions, such that bidders may choose between entering a FPA and a SPA, and bidders are risk neutral, then bidders will enter each auction with equal probability as well. When bidders are risk averse, however, the equilibrium probability of entry depends on the bidders' degree of absolute risk aversion. We distinguish between constant absolute risk aversion ($\partial r / \partial m_i = 0$), decreasing absolute risk aversion ($\partial r / \partial m_i < 0$), and increasing absolute risk aversion ($\partial r / \partial m_i > 0$).

Proposition 1. *Suppose that seller 1 selects a first-price auction ($a_1 = \text{FPA}$) and seller 2 selects a second-price auction ($a_2 = \text{SPA}$), and that bidders are risk averse ($r > 0$). The symmetric entry equilibrium is then given by*

- (i) $q^* = 0.5$, if bidders exhibit constant absolute risk aversion (CARA)
- (ii) $q^* < 0.5$, if bidders exhibit decreasing absolute risk aversion (DARA)
- (iii) $q^* > 0.5$, if bidders exhibit increasing absolute risk aversion (IARA)

where q^* defines the equilibrium probability of entering the first-price auction and $1 - q^*$ defines the equilibrium probability of entering the second-price auction.

Proof. The proof of Proposition 1 consists of two steps. Recall that the value of q^* that satisfies $z(q^*, r) = 0$ characterizes the symmetric equilibrium. In Step 1, we show that if each auction is entered with equal probability ($q = 0.5$) then $z(0.5, r)$ is equal to zero if bidders exhibit CARA, is negative if bidders exhibit DARA, and is positive if bidders exhibit IARA. In Step 2 of the proof, we demonstrate how q needs to be adjusted such that the equilibrium condition is satisfied.

Step 1: Suppose that $r > 0$ and that $q = 0.5$. From Theorem 1 of Matthews (1987, p.638) it then follows that, for $n_1 = n_2$, the ex ante expected utility in each auction is

- (i) $E[u|\text{FPA}, n_1] = E[u|\text{SPA}, n_2]$, if bidders exhibit CARA
- (ii) $E[u|\text{FPA}, n_1] < E[u|\text{SPA}, n_2]$, if bidders exhibit DARA
- (iii) $E[u|\text{FPA}, n_1] > E[u|\text{SPA}, n_2]$, if bidders exhibit IARA

This implies that, for a given $q = 0.5$, the LHS of (1) is equal to the RHS if bidders exhibit CARA, is smaller than the RHS if bidders exhibit DARA, and is larger than the RHS if bidders exhibit IARA. Hence,

- (i) $z(0.5, r) = 0$, if bidders exhibit CARA
- (ii) $z(0.5, r) < 0$, if bidders exhibit DARA
- (iii) $z(0.5, r) > 0$, if bidders exhibit IARA

Step 2: (i) of Proposition 1 follows immediately from Lemma 1. What follows here is a proof of (ii). From the proof of Lemma 1 we know that $z(q, r)$ is continuous and monotonically decreasing in q . As $z(0.5, r) < 0$ if bidders exhibit DARA, it therefore follows that q needs to decrease in order to achieve equilibrium. As a result, $q^* < 0.5$ if bidders exhibit DARA. (iii) of Proposition 1 is proven analogously. \square

Proposition 1 implies that if bidders exhibit CARA, they will enter the FPA and SPA with equal probability. However, bidders will be more likely to enter the SPA if they exhibit DARA, and will be more likely to enter the FPA if they exhibit IARA. These findings follow from the utility equivalence results from Matthews (1983, 1987), who compares auctions for risk averse bidders when the number of bidders in each auction is fixed. Risk averse bidders tend to bid more in the FPA than in the SPA, making the SPA more desirable from the bidders' perspective. At the same time, however, the payment in the SPA is a random variable, making the FPA more desirable. Matthews (1987) finds that a bidder prefers the SPA to the FPA if she exhibits DARA. Conversely, she prefers the FPA if she exhibits IARA. If the bidder exhibits CARA, she is indifferent between the two auctions.⁷ Combining these findings with the fact that the expected utility of an auction is decreasing in the number of bidders, gives us Proposition 1. For instance, suppose that bidders exhibit DARA. In this case, each bidder is only indifferent between entering a FPA and a SPA when the number of competing bidders is larger in the SPA than in the FPA. Similarly, if bidders exhibit IARA, each bidder is only indifferent between entering a FPA and a SPA when the number of competing bidders is larger in the FPA than in the SPA.

Simulations with utility functions exhibiting different degrees of absolute risk aversion show that q^* remains close to 0.5 for any r . This can be seen in Figures 1 to 3 in Section 4, which show how q^* develops when bidders exhibiting DARA get more risk averse.⁸ It seems that even though bidders may highly prefer one auction over the other, there are negative externalities from other bidders entering the auction. This latter effect seems to be rather strong, causing q^* to remain close to 0.5 even when bidders have a strong preference for one of the auctions.

4. Auction selection

In this section, we use the insights obtained in Section 3 to evaluate the sellers' decisions in Stage 1 of our game. Recall that there are two sellers, who each offer one unit of a homogeneous good in either a FPA or a SPA. With a slight abuse of notation, we will from now on define q as the entry probability into the FPA and $1 - q$ as the entry probability into the SPA. The expected revenues are then given by

$$E[R_{FPA}] = \sum_{n_1=0}^N \binom{N}{n_1} (q)^{n_1} (1-q)^{N-n_1} R_{FPA}(n_1, r)$$

$$E[R_{SPA}] = \sum_{n_1=0}^N \binom{N}{n_1} (1-q)^{n_1} (q)^{N-n_1} R_{SPA}(n_1)$$

⁷To the best of our knowledge, there is no easy intuitive explanation for Matthews's (1987) finding. Rather, it is based on the mathematical fact that if a bidder's utility is increasing in her value, such that $\frac{\partial u}{\partial v_i} > 0$, and she exhibits DARA (CARA) (IARA), then $\frac{\partial u}{\partial v_i}$ is strictly convex (linear) (strictly concave) in u (for details, see Lemma 1 by Maskin and Riley (1984, p.1479)). By using this fact and by writing the expected utilities in the FPA and SPA as functions of the winning bidders' respective payments, Matthews proves that the certainty equivalent of the random payment in the SPA is smaller than (equal to) (larger than) the payment in the FPA if bidders exhibit DARA (CARA) (IARA).

⁸As decreasing absolute risk aversion is implied by constant relative risk aversion (CRRA), we focus on the effect of different levels of CRRA in our simulations underlying Figures 1 to 3.

where $R_{FPA}(n_1, r)$ is the seller's ex ante expected revenue when the FPA is entered by n_1 bidders who have risk parameter r . It represents the expected payment made by the highest of n_1 bidders. Similarly, $R_{SPA}(n_1)$ is the seller's ex ante expected revenue when the SPA is entered by n_1 bidders. The ex ante expected revenues of both auctions are increasing in the number of bidders n_1 (e.g. Kagel and Levin, 1993).

The revenue equivalence theorem states that the ex ante expected revenue from the FPA equals that of the SPA if bidders are risk neutral, that is, $R_{FPA}(n_1, 0) = R_{SPA}(n_1)$ (Vickrey, 1961). Recall that, in equilibrium, bidders enter each auction with equal probability ($q^* = 0.5$) if they are risk neutral (see Lemma 2). Hence, it immediately follows that $E[R_{FPA}] = E[R_{SPA}]$ if bidders are risk neutral, and therefore competing sellers will be indifferent between selecting the FPA and the SPA.

If bidders are risk averse, the situation is more complex. Whereas the equilibrium bidding strategy in the SPA is insensitive to changes in risk attitudes, the equilibrium bidding strategy in the FPA is increasing in risk aversion (e.g. Riley and Samuelson, 1981; Maskin and Riley, 1984; Cox et al., 1985, 1988). As a result, the ex ante expected revenue of the FPA is larger than that of the SPA if bidders are risk averse, that is, $R_{FPA}(n_1, r) > R_{SPA}(n_1)$ for $r > 0$. Given our results from Section 3, this implies the following for the expected revenues.

Lemma 3. *Suppose that seller 1 selects a first-price auction ($a_1 = FPA$) and seller 2 selects a second-price auction ($a_2 = SPA$), and that bidders are risk averse ($r > 0$), exhibit nondecreasing absolute risk aversion and follow the symmetric entry equilibrium defined in Proposition 1. The first-price auction then yields more expected revenue than the second-price auction.*

Proof. Proposition 1 shows that $q^* \geq 0.5$ if bidders exhibit CARA or IARA, where q^* defines the equilibrium probability of entering the FPA and $1 - q^*$ defines the equilibrium probability of entering the SPA. This permits direct comparison of expected revenues.

$$\begin{aligned} E[R_{FPA}] &= \sum_{n_1=0}^N \binom{N}{n_1} (q^*)^{n_1} (1 - q^*)^{N-n_1} R_{FPA}(n_1, r) \\ &> \sum_{n_1=0}^N \binom{N}{n_1} (q^*)^{n_1} (1 - q^*)^{N-n_1} R_{SPA}(n_1) \\ &\geq \sum_{n_2=0}^N \binom{N}{n_2} (1 - q^*)^{n_2} (q^*)^{N-n_2} R_{SPA}(n_2) = E[R_{SPA}] \end{aligned}$$

The strict inequality is based on the fact that $R_{FPA}(n_1, r) > R_{SPA}(n_1)$ for $r > 0$. To prove that the second inequality holds we rewrite the expected revenues as

$$\begin{aligned} E[R_{FPA}] &= \sum_{n_1=0}^N p_{n_1:N}(q^*) R_{FPA}(n_1, r) \\ E[R_{SPA}] &= \sum_{n_2=0}^N p_{n_2:N}(q^*) R_{SPA}(n_2) \end{aligned}$$

where $p_{n_1:N}(q^*) = \binom{N}{n_1} (q^*)^{n_1} (1 - q^*)^{N-n_1}$ and $p_{n_2:N}(q^*) = \binom{N}{n_2} (1 - q^*)^{n_2} (q^*)^{N-n_2}$. We can show that $E[R_{FPA}]$ is continuous and monotonically increasing in q (see Lemma A2 in Appendix A) and that $E[R_{SPA}]$ is continuous and monotonically decreasing in q (see Lemma A3 in Appendix A). As $p_{n_1:N}(q) = p_{n_2:N}(q)$ for $q = 0.5$, it then follows that $p_{n_1:N}(q) > p_{n_2:N}(q)$ for any $q > 0.5$, and $p_{n_1:N}(q) < p_{n_2:N}(q)$ for any $q < 0.5$. Since $q^* \geq 0.5$ if bidders exhibit CARA or IARA (see Proposition 1) and since $R_{SPA}(n_2)$ is increasing in n_2 , the second inequality must hold. This concludes the proof of Lemma 3. \square

If competing sellers offer their goods in both FPAs and SPAs, and risk averse bidders endogenously enter one of the auctions, then each bidder is at least as likely to enter the FPA as she is likely to enter the SPA (see Proposition 1). This finding, combined with the familiar ranking of ex ante expected revenues for risk averse bidders, gives us Lemma 3. Our finding also implies that DARA is a necessary condition for the traditional revenue ranking to reverse. After all, if bidders exhibit DARA, they prefer the SPA over the FPA, which makes them more likely to enter the SPA. Only if sufficiently many bidders enter the SPA, the initial advantage of the FPA may be overcome.

Table 1: Payoffs of the auction selection game

Seller 1 \ Seller 2	FPA	SPA
FPA	$\sum_{n_1=0}^N \binom{N}{n_1} 0.5^{n_1} 0.5^{N-n_1} R_{FPA}(n_1, r)$	$\sum_{n_1=0}^N \binom{N}{n_1} (q^*)^{n_1} (1 - q^*)^{N-n_1} R_{FPA}(n_1, r)$
SPA	$\sum_{n_1=0}^N \binom{N}{n_1} (1 - q^*)^{n_1} (q^*)^{N-n_1} R_{SPA}(n_1)$	$\sum_{n_1=0}^N \binom{N}{n_1} 0.5^{n_1} 0.5^{N-n_1} R_{SPA}(n_1)$

We now turn to the auction selection game, where we study which auctions competing sellers select when bidders are risk averse and endogenously enter auctions. Table 1 gives the payoffs of the auction selection game of seller 1 (the row player); the payoffs of seller 2 are symmetric. From Lemma 2 we know that $q^* = 0.5$ for $a_1 = a_2$. Following from the revenue ranking for risk averse bidders, the strategy combination (FPA, FPA) dominates (SPA, SPA) in terms of total payoffs. The ranking of the other strategy combinations is influenced by the degree of absolute risk aversion of the bidders, as it crucially depends on the value of the equilibrium probability of entry, q^* .

Proposition 2. *Suppose that two competing sellers choose between selecting a first-price auction and a second-price auction, and that bidders are risk averse ($r > 0$), exhibit nondecreasing absolute risk aversion and follow the symmetric entry equilibrium defined in Proposition 1. Then each seller has a dominant strategy to select the first-price auction.*

Proof. This proof makes use of the mutual best response property of a Nash equilibrium. By Proposition 1, $q^* \geq 0.5$ if bidders exhibit CARA or IARA. Further recall that $R_{FPA}(n_l, r) > R_{SPA}(n_l)$, that $E[R_{FPA}]$ is continuous and monotonically increasing in q , and that $E[R_{SPA}]$ is continuous and monotonically decreasing in q (for the latter two findings, see Lemmata A2 and A3 in Appendix A). To determine the best response for seller $l = \{1, 2\}$, first suppose that seller $-l$ selects FPA. Then by the above it follows that

$$0.5^N \sum_{n_l=0}^N \binom{N}{n_l} R_{FPA}(n_l, r) > \sum_{n_l=0}^N \binom{N}{n_l} (1 - q^*)^{n_l} (q^*)^{N-n_l} R_{SPA}(n_l)$$

Similarly, suppose that seller $-l$ selects SPA, then

$$\sum_{n_l=0}^N \binom{N}{n_l} (q^*)^{n_l} (1 - q^*)^{N-n_l} R_{FPA}(n_l, r) > 0.5^N \sum_{n_l=0}^N \binom{N}{n_l} R_{SPA}(n_l)$$

This implies that selecting FPA is a dominant strategy for seller $l = \{1, 2\}$ and concludes the proof of Proposition 2. \square

Proposition 2 implies that if bidders exhibit nondecreasing absolute risk aversion, all competing sellers select a FPA. This follows naturally, as in these cases the FPA is ex ante (weakly) preferred to the SPA by both sellers and bidders. If bidders exhibit DARA, however, two opposing effects occur. On the one hand,

the FPA generates more ex ante expected revenue than the SPA if bidders are risk averse. On the other hand, if bidders exhibit DARA, they are more likely to enter the SPA than the FPA, that is, $q^* < 0.5$ by Proposition 1. Proposition 2 implies that DARA is a necessary condition for any equilibrium other than (FPA, FPA) to exist, but it is by itself not sufficient. In the remainder of this section, we demonstrate by example that if bidders exhibit DARA, other equilibria may exist in which sellers select SPAs as well.

4.1. An example of auction selection with DARA bidders

Consider the following example, where bidder i has a utility function of the form $u(m_i) = m_i^{(1-\rho)}$, where m_i represents a bidder's payoff and $\rho \in [0, 1)$ represents the coefficient of constant relative risk aversion (CRRA).⁹ Recall that if bidder i wins the auction, her payoff (m_i) is equal to her private value (v_i) minus her payment (p_i). If bidder i loses the auction, her payoff is equal to zero. Values are distributed according to $F(v) = v^\alpha$ for $v \in [0, 1]$, where $\alpha \geq 1$ and takes integer values only. Note that values are uniformly distributed if $\alpha = 1$. An increase in α represents an increase in the skewness of the distribution of values such that higher values are drawn with larger probability. In this case, the ex ante expected revenues are given by

$$R_{FPA}(n_l, r) = \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} \frac{\alpha n_l}{\alpha n_l + 1} \quad (2)$$

$$R_{SPA}(n_l) = \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1} \frac{\alpha n_l}{\alpha n_l + 1} \quad (3)$$

The bidders' ex ante expected utilities in the auctions are given by

$$E[u|FPA, n_l] = \frac{\alpha}{\alpha n_l + 1 - \rho} \left(\frac{1 - \rho}{\alpha(n_l - 1) + 1 - \rho} \right)^{1-\rho} \quad (4)$$

$$E[u|SPA, n_l] = \frac{\alpha}{\alpha n_l + 1 - \rho} \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!} \quad (5)$$

where $(\alpha(n_l - 1) + 1 - \rho)! \equiv \prod_{i=1}^{\alpha(n_l - 1)} (i + 1 - \rho)$. The derivations of these results can be found in Appendix B.¹⁰

To analyze which auctions are selected by competing sellers, we use (4) and (5) to compute the equilibrium probability of entry, q^* , and use (2) and (3) to compute \underline{q} and \bar{q} . Let \underline{q} be defined as the probability of entry for which seller $l = \{1, 2\}$ is indifferent between selecting the FPA and the SPA given that seller $-l$ offers a FPA.

$$0.5^N \sum_{n_l=0}^N \binom{N}{n_l} R_{FPA}(n_l, r) = \sum_{n_l=0}^N \binom{N}{n_l} (1 - \underline{q})^{n_l} (\underline{q})^{N-n_l} R_{SPA}(n_l) \quad (6)$$

Similarly, let \bar{q} be defined as the probability of entry for which seller $l = \{1, 2\}$ is indifferent between selecting the FPA and the SPA given that seller $-l$ offers a SPA.

$$\sum_{n_l=0}^N \binom{N}{n_l} (\bar{q})^{n_l} (1 - \bar{q})^{N-n_l} R_{FPA}(n_l, r) = 0.5^N \sum_{n_l=0}^N \binom{N}{n_l} R_{SPA}(n_l) \quad (7)$$

⁹For simplicity, we have chosen to present here the simulations for one of the simplest and most often used utility functions in economics: the power utility function for positive powers. However, qualitatively similar results can be obtained when using a more general utility function, for instance, one exhibiting hyperbolic absolute risk aversion. For a discussion of the characteristics of the power utility function, see Wakker (2008).

¹⁰An alternative way of formulating $E[u|SPA, n_l]$ is as a function of the gamma function, Γ . In this case, it is given by

$$E[u|SPA, n_l] = \frac{\alpha}{\alpha n_l + 1 - \rho} \frac{\Gamma(\alpha(n_l - 1) + 1)\Gamma(2 - \rho)}{\Gamma(\alpha(n_l - 1) + 2 - \rho)}$$

Note that because $R_{FPA}(n_I, r) > R_{SPA}(n_I)$ for $r > 0$, and because $E[R_{FPA}]$ is continuous and monotonically increasing in q (see Lemma A2 in Appendix A), the LHS of (7) will be larger than the RHS for any $q \geq 0.5$. Likewise, because $E[R_{SPA}]$ is continuous and monotonically decreasing in q (see Lemma A3 in Appendix A), the LHS of (6) will be larger than the RHS for any $q \geq 0.5$. Therefore, both \underline{q} and \bar{q} will be strictly below 0.5.

Figure 1 illustrates the values of q^* , \underline{q} and \bar{q} for different values of α and ρ , and for $N = 4$. The numbered regions in Figure 1 correspond to different equilibrium outcomes. In region I, where $q > \underline{q}, \bar{q}$, sellers have a dominant strategy to select the FPA. As a result, in this region there is a unique Nash equilibrium and it is given by the strategy combination (FPA, FPA). In region II, where $q < \underline{q}, \bar{q}$, the unique Nash equilibrium is

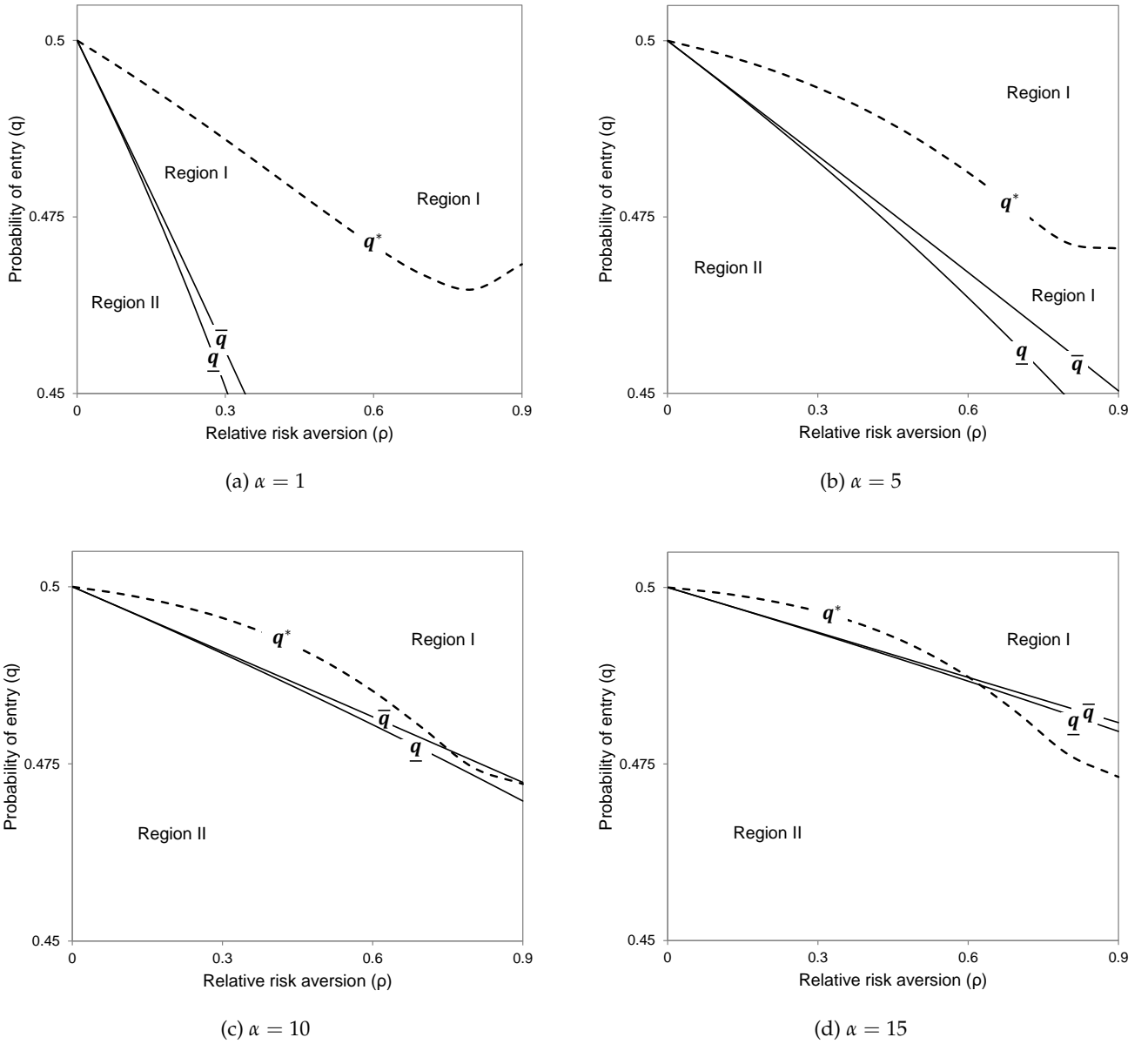


Figure 1: Effect of the distribution of values on auction selection with CRRA bidders (where $F(v) = v^\alpha$ and $N = 4$)

given by (SPA, SPA). In region III (visible for some parameter values in Figure 1 but not explicitly indicated), it is the case that $\underline{q} < \bar{q} < \bar{q}$. As $E[R_{FPA}]$ is increasing in q and $E[R_{SPA}]$ is decreasing in q , it follows that in this case, the auction selection game is in fact a coordination game. The Nash equilibria are then given by (FPA, FPA), (SPA, SPA) and one involving mixed strategies.¹¹

Figure 1a shows that, when values are uniformly distributed, q^* remains above \underline{q} and \bar{q} for any $\rho \in (0, 1)$. This implies that sellers have a dominant strategy to select the FPA. However, as the distribution function becomes more skewed (α becomes larger), \underline{q} and \bar{q} shift upwards, leading to an increase in region II at the expense of region I. As a result, we find that if the distribution of values is sufficiently skewed and bidders are sufficiently risk averse then q^* also moves through regions II and III (see Figures 1c and 1d), such that in equilibrium both sellers could also end up selecting SPAs. Our finding is analogous to that of Smith and Levin (1996), who show, in a model where bidders can choose whether or not to enter an auction at an entry cost, that the traditional revenue ranking for risk averse bidders can be reversed if the distribution of values is sufficiently skewed. The reason for these results is that an increase in α reduces the variance in payments generated in the SPA and thereby decreases the difference in ex ante expected revenues between the FPA and SPA. This can immediately be seen from (2) and (3), where an increase in α leads to a relatively larger change in the ex ante expected revenue for the SPA than for the FPA.

Smith and Levin (1996) suspect that increasing the number of bidders (N) affects the revenue ranking between the FPA and SPA in a similar way as increasing the skewness of the distribution does (α). They therefore “conjecture that SPA would tend to be favored by the seller more often in markets with many potential bidders than in markets with few” (Smith and Levin, 1996, p.558). We find that this does not hold for our setting. Rather, we find that increasing the number of bidders decreases both \underline{q} and \bar{q} , thereby making it less likely that the dominance of FPA is overthrown. Figures 2 and 3 show the effect of increasing N to 6 and 9 when values are uniformly distributed ($\alpha = 1$) and when the distribution of values is rather skewed ($\alpha = 15$), respectively. This finding extends to larger N as well.

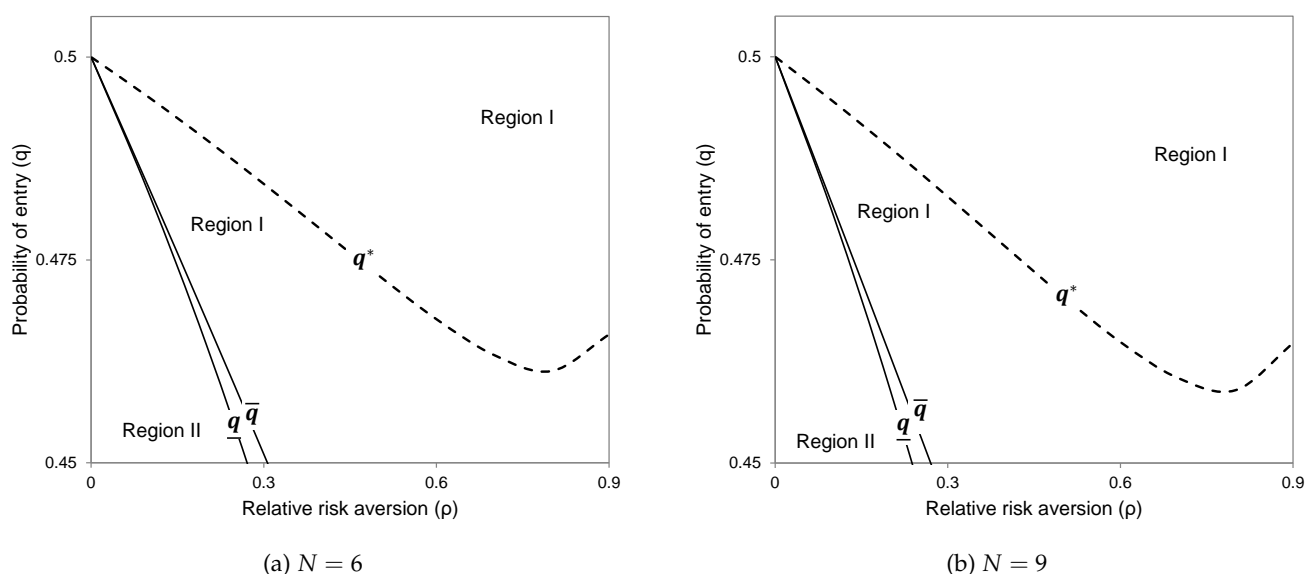


Figure 2: Effect of N on auction selection with CRRA bidders and a uniform distribution of values (where $F(v) = v^\alpha$ and $\alpha = 1$)

¹¹Note that there may exist a fourth possible equilibrium outcome, i.e., where $\underline{q} > \bar{q} > \bar{q}$. In this case, the auction selection game is in fact an anti-coordination game, such that the resulting Nash equilibria are given by (FPA, SPA), (SPA, FPA) and an equilibrium

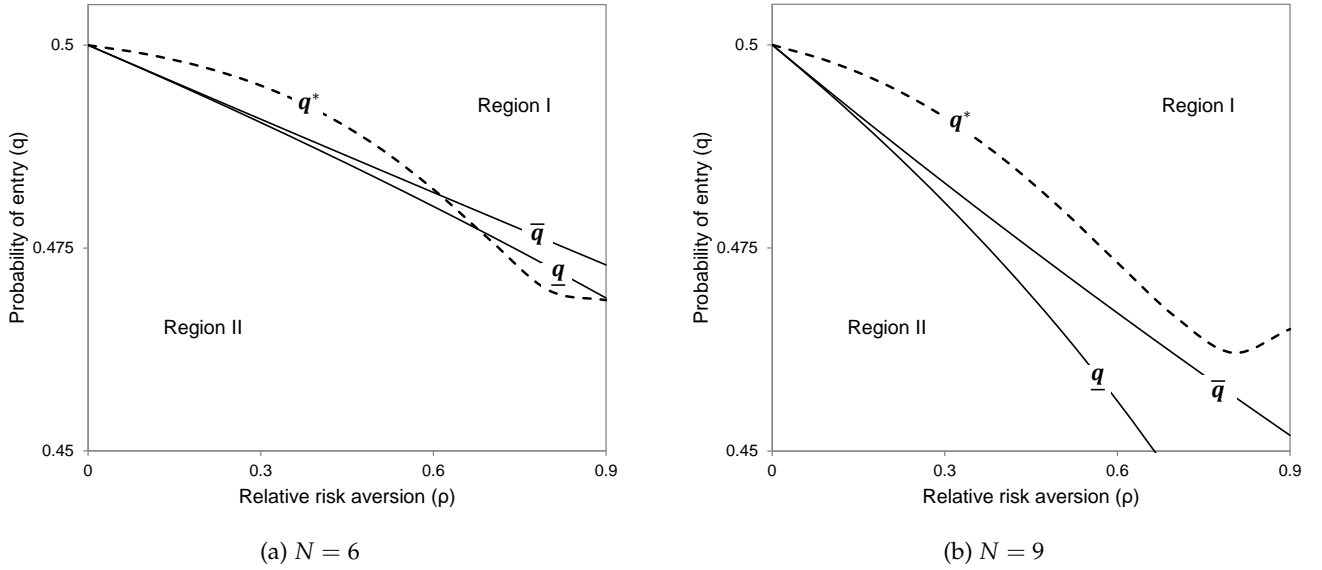


Figure 3: Effect of N on auction selection with CRRA bidders and a skewed distribution of values (where $F(v) = v^\alpha$ and $\alpha = 15$)

5. Extensions

In this section, we consider extensions where bidders have heterogeneous risk attitudes (Section 5.1), where both goods are owned by the same seller (Section 5.2) and where the number of sellers is increased to $M > 2$ (Section 5.3).

5.1. Heterogeneous risk attitudes

Our model assumes that bidders are ex ante homogeneous, meaning that they all maximize the same utility function and do not know their own value for the good before deciding which auction to enter. In this sense, bidders' entry decisions are modeled as a game of complete information. In Section 3, we show that this results in a mixed strategy Nash equilibrium, where each bidder enters one auction with probability q^* and enters the other auction with probability $1 - q^*$. By the purification theorem of Harsanyi (1973), our mixed strategy Nash equilibrium can be interpreted as a pure strategy Bayesian Nash equilibrium of an entry game with incomplete information, for instance, one where bidders have heterogeneous risk attitudes.

Like before, let us assume that q^* denotes the equilibrium probability of entry into the FPA and $1 - q^*$ denotes the equilibrium probability of entry into the SPA. Recall that if bidders are homogeneously risk neutral ($r = 0$), the equilibrium probability of entry equals $q^* = 0.5$ (see Lemma 2). If bidders are homogeneously risk averse ($r > 0$), then $q^* = 0.5$ if bidders exhibit CARA, $q^* < 0.5$ if bidders exhibit DARA and $q^* > 0.5$ if bidders exhibit IARA (see Proposition 1). This suggests that for homogeneous bidders a range of risk parameters around risk neutrality exists, such that the equilibrium probability of entry q^* is constant in r if bidders exhibit CARA, is decreasing in r if bidders exhibit DARA, and is increasing in r if bidders exhibit IARA. Let us denote this range of risk aversion parameters by $r \in [0, r^*]$. From Figures 1 to 3, it can be seen that if homogeneous bidders exhibit DARA, q^* is indeed initially decreasing in the risk parameter (ρ), but becomes increasing as bidders get very risk averse ($\rho > 0.8$).

Now suppose that bidders are heterogeneous, i.e., they all maximize the same utility function exhibiting either decreasing, constant, or increasing absolute risk aversion, but have different risk parameters

involving mixed strategies. While we do not find any evidence for cases where $q > \bar{q}$ in our simulations, we cannot rule out that such cases exist for certain distribution functions or utility functions.

$r_i \in [0, r^*]$, which are independently drawn from a distribution function $G(r)$. Before entry, each bidder knows her own risk parameter (r_i) and the distribution of other risk parameters ($G(r)$). We may then follow the approach of Pevnitskaya (2004) and find that there exists a self-selection effect when heterogeneous bidders decide between entering the FPA and SPA. More specifically, as q^* is decreasing in r for homogeneous bidders exhibiting DARA, it can be shown that for heterogeneous bidders exhibiting DARA there exists a cut-off point r' such that the more risk tolerant bidders ($r < r'$) enter the FPA and the more risk averse bidders ($r > r'$) enter the SPA. For heterogeneous bidders exhibiting IARA the self-selection effect is reversed: more risk tolerant bidders enter the SPA and more risk averse bidders enter the FPA. For the case of heterogeneous bidders exhibiting CARA anything goes.

In an experimental study, where bidders choose between entering an English auction and a FPA, Ivanova-Stenzel and Salmon (2008a) find no support for a selection effect based on risk aversion. To analyze the self-selection effect, the authors use the amount of overbidding as a measure of risk aversion. More specifically, they assume that bidders preferences can be described by $u(m_i) = m_i^{1-\rho}$, which results in an equilibrium bidding strategy $b(v) = ((n_l - 1)/(n_l - \rho))v$ when values are uniformly distributed between $[0, 1]$. They find that neither risk aversion nor, more generally, the degree of overbidding has a statistically significant effect on bidders' entry decisions. At first sight, this does not seem to be in line with our findings. It would therefore be interesting to further explore what drives the differences in results between theory and experiment. Also interesting, but beyond the scope of the present study, would be to investigate what bidder heterogeneity implies for the sellers' decisions in the auction selection stage.

5.2. Monopoly

Recently, some sellers have started offering a single good in multiple selling mechanisms at the same time. A Dutch travel agency,¹² for instance, sells holidays through ascending auctions, next to selling them at a posted price. In the United Kingdom, one company¹³ offers its customers two auction formats from which they may choose: FPAs and lowest unique bid auctions. This suggests that the mechanism through which goods are sold has become the subject of versioning. Therefore, we extend our model to a monopoly setting. Consider a monopolist who sells two units of a homogeneous good and decides to offer these in two simultaneous auctions. He can either choose to offer two FPAs, two SPAs, or a combination of a FPA and a SPA. The monopolist's objective is to maximize the sum of expected revenues of each strategy profile listed in Table 1. Alternatively, the monopoly setting can be interpreted as representing the auction selection decisions of competing sellers when they collude.

Proposition 3. *Suppose that a monopolist sells his goods in two simultaneous auctions and chooses between first-price and second-price auctions, and that bidders are risk averse ($r > 0$) and follow the symmetric entry equilibrium defined in Proposition 1. Then there exists a range of risk parameters around risk neutrality such that a monopolist prefers to offer both units in first-price auctions.*

Proof. Recall that $q^* = 0.5$ if $a_1 = a_2$ (see Lemma 2). Additionally, recall that the traditional revenue ranking implies that $R_{FPA}(n_l, r) > R_{SPA}(n_l)$ for $r > 0$. It therefore follows immediately that the sum of expected revenues of (FPA, FPA) is greater than that of (SPA, SPA). Consequently, to prove Proposition 3, it suffices to show that the sum of expected revenues of (FPA, FPA) is greater than that of (FPA, SPA). The sum of expected revenues of offering both a FPA and a SPA is given by

$$\sum_{n_l=0}^N \binom{N}{n_l} (q^*)^{n_l} (1 - q^*)^{N-n_l} R_{FPA}(n_l, r) + \sum_{n_l=0}^N \binom{N}{n_l} (1 - q^*)^{n_l} (q^*)^{N-n_l} R_{SPA}(n_l)$$

¹²Emesa.nl

¹³Auctionair.co.uk

The sum of expected revenues of two FPAs is given by

$$0.5^N \sum_{n_l=0}^N \binom{N}{n_l} R_{FPA}(n_l, r) + 0.5^N \sum_{n_l=0}^N \binom{N}{n_l} R_{FPA}(n_l, r)$$

To prove by contradiction, assume that the sum of expected revenues of (FPA, SPA) is at least as large as that of (FPA, FPA).

$$\sum_{n_l=0}^N \binom{N}{n_l} \left\{ [(q^*)^{n_l} (1 - q^*)^{N-n_l} - 2 * 0.5^N] R_{FPA}(n_l, r) + (1 - q^*)^{n_l} (q^*)^{N-n_l} R_{SPA}(n_l) \right\} \geq 0 \quad (8)$$

By the revenue equivalence theorem, $R_{FPA}(n_l, 0) = R_{SPA}(n_l)$, and by Lemma 2, $q^* = 0.5$ for $r = 0$. At risk neutrality ($r = 0$), the sum of expected revenues of (FPA, FPA) must be equal to that of (FPA, SPA). Consequently, it suffices to show that at $r = 0$ the derivative of (8) with respect to r is nonnegative. Differentiating (8) with respect to r produces the following equation.

$$\sum_{n_l=0}^N \binom{N}{n_l} \left\{ [(q^*)^{n_l} (1 - q^*)^{N-n_l} - 2 * 0.5^N] \frac{\partial R_{FPA}(n_l, r)}{\partial r} + (q^*)^{n_l-1} (1 - q^*)^{N-n_l-1} [n_l - q^* N] \frac{dq^*}{dr} R_{FPA}(n_l, r) + (1 - q^*)^{n_l-1} (q^*)^{N-n_l-1} [(1 - q^*) N - n_l] \frac{dq^*}{dr} R_{SPA}(n_l) \right\} \geq 0$$

We now evaluate this at $r = 0$, which by Lemma 2 implies $q^* = 0.5$.

$$\sum_{n_l=0}^N \binom{N}{n_l} \left\{ -0.5^N \frac{\partial R_{FPA}(n_l, r)}{\partial r} + 0.5^{N-2} [n_l - 0.5N] \frac{dq^*}{dr} R_{FPA}(n_l, 0) + 0.5^{N-2} [0.5N - n_l] \frac{dq^*}{dr} R_{SPA}(n_l) \right\} \geq 0$$

where the last two terms cancel out as $R_{FPA}(n_l, 0) = R_{SPA}(n_l)$ and where $\frac{\partial R_{FPA}(n_l, 0)}{\partial r} > 0$. As a result, the equation above is strictly negative, contradicting our assumption. This concludes the proof of Proposition 3. \square

Proposition 3 states that, for some range around risk neutrality, a monopolist prefers to offer two FPAs to offering them in different auctions or in SPAs. This result is independent of whether bidders exhibit CARA, DARA or IARA. In case of CARA, however, we show that the result is more general.

Corollary 1. *Suppose that that a monopolist sells his goods in two simultaneous auctions and chooses between first-price and second-price auctions, and that bidders are risk averse ($r > 0$), exhibit constant absolute risk aversion, and follow the symmetric entry equilibrium defined in Proposition 1. Then a monopolist prefers to offer both units in first-price auctions to offering them in a first-price and second-price auction, which is preferred to offering them in second-price auctions.*

Proof. By Proposition 1, we know that with CARA bidders $q^* = 0.5$ for every r . As $R_{FPA}(n_l, r) > R_{SPA}(n_l)$ for $r > 0$, it follows immediately that the sum of expected revenues from (FPA, FPA) is greater than that of (FPA, SPA), which is in turn greater than that of (SPA, SPA). \square

Simulations for utility functions exhibiting different degrees of absolute risk aversion consistently show that a monopolist prefers to select only FPAs. We therefore conjecture that Corollary 1 holds as well for bidders exhibiting DARA or IARA. Our findings are consistent with traditional revenue ranking theorems, but seem less consistent with practice in online auctions. Whereas our results indicate that it is not profitable to use auction design as a means of versioning, this is exactly what happens on the Internet. Perhaps such versioning by monopolists can only be explained when bidders have heterogeneous or non-standard preferences. Future research might therefore consider heterogeneous risk averse (and risk seeking) bidders or take into account behavioral assumptions such as reference-dependent preferences and competitiveness. Taking into account more sophisticated assumptions might better explain bidders' entry decisions and, hence, the form that auction versioning by monopolists takes.

5.3. $M > 2$ sellers

Our results can easily be extended to a market with $M \geq 2$ competing sellers. From Lemma 2, it immediately follows that if all sellers offer the same auction or if bidders are risk neutral ($r = 0$), each bidder enters each auction $a_l = 1, 2, \dots, M$ with probability $q_l^* = (1/M)$. Now suppose that seller l offers a FPA and all other $M - 1$ sellers offer SPAs, and that bidders are risk averse ($r > 0$). Then by Proposition 1, the equilibrium probability of entry equals $q_l^* = q_{-l}^* = (1/M)$ if bidders exhibit CARA. Likewise, if bidders exhibit DARA, $q_l^* < (1/M)$ and $q_{-l}^* > (1/M)$, and if bidders exhibit IARA, $q_l^* > (1/M)$ and $q_{-l}^* < (1/M)$. In the auction selection game, sellers will continue to have a dominant strategy to select FPAs if bidders exhibit CARA or IARA.

6. Conclusion

The main objective of this paper is to investigate which auctions are selected by competing sellers when they may choose between first-price and second-price auctions and when risk averse bidders endogenously enter one of the auctions. We construct a three-stage game in which two units of a homogenous good are offered simultaneously to a group of N homogeneously risk averse bidders. At Stage 1, the sellers each select an auction; at Stage 2, each bidder learns which auctions have been selected and decides to enter one of the auctions; finally, at Stage 3, the auctions are conducted.

Our key findings can be summarized along two lines. First, we show that when bidders may choose between entering the first-price and second-price auction, then a symmetric equilibrium exists involving mixed strategies, where the mixing probabilities depend on the bidders' degree of absolute risk aversion. If bidders exhibit risk neutrality or constant absolute risk aversion, they will enter each auction with equal probability. If bidders exhibit decreasing absolute risk aversion, however, they will enter the second-price auction with greater likelihood, and if bidders exhibit increasing absolute risk aversion, they will enter the first-price auction with greater likelihood. Second, we find that if bidders exhibit nondecreasing absolute risk aversion, competing sellers have a dominant strategy to select first-price auctions. We demonstrate by example that if bidders exhibit decreasing absolute risk aversion, sellers may also select second-price auctions if the distribution of private values is sufficiently skewed.

Whereas traditional revenue ranking theorems predict that competing sellers should prefer the first-price auction when bidders are risk averse, in reality most sellers seem to offer English auctions, which are strategically equivalent to second-price auctions. Our analysis suggests that this could be explained by the presence of decreasing absolute risk aversion. Additionally, even though experimental studies often assume that values are uniformly distributed, it is possible that in many real-world auctions values actually follow a more skewed distribution. Future research might further explore this, both experimentally and empirically.

In the context of online auctions, it would also be interesting to explore to which extent our findings depend on the assumption that bidders know how many other bidders actually enter each auction. After all, on the Internet, bidders may not be aware of how many competing bidders participate in an auction. Matthews (1987) shows that the preference rankings for risk averse bidders can be extended to a setting where the number of bidders participating in each auction is concealed. We therefore conjecture that in such a setting, there exists an entry equilibrium analogous to the one we find in this paper. Future research may consider the effects of concealing the number of competing bidders on bidders' entry decisions and its implications for the auction selection decisions of competing sellers.

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Appendix

A. Additions and proofs

Lemma A1. *The RHS of (1) is continuous and monotonically increasing in q for a given r .*

Proof. This proof follows the same line of reasoning as the proof of Lemma 1 by Pevnitskaya (2004, p.6). For simplicity we rewrite the RHS of (1) as

$$\sum_{n_2=1}^N p_{n_2-1:N-1}(q) * x_{n_2}$$

where $p_{n_2-1:N-1}(q) = \binom{N-1}{n_2-1}(1-q)^{n_2-1}q^{N-n_2}$ and $x_{n_2} = E[u|a_2, n_2]$. For a given risk parameter, r , among N elements of the sum only the expression $(1-q)^{n_2-1}q^{N-n_2}$ is a function of q . Since it is continuous in q , the sum of N elements is continuous in q as well. To show that the RHS of (1) is increasing in q for a given r , we therefore only need to prove that

$$\sum_{n_2=1}^N [p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2)]x_{n_2} > 0 \quad \text{for } q_1 > q_2$$

To prove by contradiction, assume that

$$\sum_{n_2=1}^N [p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2)]x_{n_2} \leq 0 \quad \text{for } q_1 > q_2$$

From the binomial density function properties we know that $p_{n_2-1:N-1}(q_1) > p_{n_2-1:N-1}(q_2)$ for small n_2 , and vice versa for large n_2 . Therefore, there exists some η , such that $[p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2)]x_{n_2} \geq 0$ for any $n_2 \leq \eta$, and $[p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2)]x_{n_2} < 0$ for any $n_2 > \eta$. The equation above can be rewritten as follows.

$$\begin{aligned} & \sum_{n_2=1}^{\eta} [p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2)]x_{n_2} \\ & \leq \sum_{n_2=\eta+1}^N [-(p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2))]x_{n_2} \end{aligned}$$

Since x_{n_2} is decreasing in n_2 , we further have

$$\begin{aligned} & \sum_{n_2=1}^{\eta} [p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2)]x_{n_{\eta+1}} < LHS \\ & \leq RHS \leq \sum_{n_2=\eta+1}^N [-(p_{n_2-1:N-1}(q_1) - p_{n_2-1:N-1}(q_2))]x_{n_{\eta+1}} \end{aligned}$$

This implies the following:

$$\begin{aligned} & \sum_{n_2=1}^{\eta} p_{n_2-1:N-1}(q_1) - \sum_{n_2=1}^{\eta} p_{n_2-1:N-1}(q_2) \\ & < \sum_{n_2=\eta+1}^N p_{n_2-1:N-1}(q_2) - \sum_{n_2=\eta+1}^N p_{n_2-1:N-1}(q_1) \end{aligned}$$

This can be rewritten as follows:

$$\sum_{n_2=1}^N p_{n_2-1:N-1}(q_1) < \sum_{n_2=1}^N p_{n_2-1:N-1}(q_2)$$

$$1 < 1$$

which is a contradiction. Therefore, the assumption does not hold and Lemma A1 is proven. \square

Lemma A2. $E[R_{FPA}]$ is continuous and monotonically increasing in q for a given r .

Proof. Recall that the expected revenue of the FPA is given by

$$E[R_{FPA}] = \sum_{n_1=0}^N p_{n_1:N}(q) R_{FPA}(n_1, r)$$

where $p_{n_1:N}(q) = \binom{N}{n_1} (q)^{n_1} (1-q)^{N-n_1}$. For a given risk parameter, r , among N elements of the sum only the expression $(q)^{n_1} (1-q)^{N-n_1}$ is a function of q . Since it is continuous in q , then the sum of N elements is continuous as well. To show that Π_{FPA} is increasing in q for a given r , we only need to prove that

$$\sum_{n_1=0}^N [p_{n_1:N}(q_1) - p_{n_1:N}(q_2)] R_{FPA}(n_1, r) > 0 \quad \text{for } q_1 > q_2$$

To prove by contradiction, assume that

$$\sum_{n_1=0}^N [p_{n_1:N}(q_1) - p_{n_1:N}(q_2)] R_{FPA}(n_1, r) \leq 0 \quad \text{for } q_1 > q_2$$

From the binomial density function properties we know that $p_{n_1:N}(q_1) < p_{n_1:N}(q_2)$ for small n_1 , and vice versa for large n_1 . Therefore, there exists some η , such that $[p_{n_1:N}(q_1) - p_{n_1:N}(q_2)] R_{FPA}(n_1, r) \leq 0$ for any $n_1 \leq \eta$, and $[p_{n_1:N}(q_1) - p_{n_1:N}(q_2)] R_{FPA}(n_1, r) > 0$ for any $n_1 > \eta$. The equation above can be rewritten as follows.

$$\sum_{n_1=0}^{\eta} [p_{n_1:N}(q_1) - p_{n_1:N}(q_2)] R_{FPA}(n_1, r)$$

$$\leq \sum_{n_1=\eta+1}^N [-(p_{n_1:N}(q_1) - p_{n_1:N}(q_2))] R_{FPA}(n_1, r)$$

Since $R_{FPA}(n_1, r)$ is increasing in n_1 , we further have

$$\sum_{n_1=0}^{\eta} [p_{n_1:N}(q_1) - p_{n_1:N}(q_2)] R_{FPA}(n_{\eta+1}, r) < LHS$$

$$\leq RHS \leq \sum_{n_1=\eta+1}^N [-(p_{n_1:N}(q_1) - p_{n_1:N}(q_2))] R_{FPA}(n_{\eta+1}, r)$$

This implies the following:

$$\begin{aligned} \sum_{n_1=0}^{\eta} p_{n_1:N}(q_1) - \sum_{n_1=0}^{\eta} p_{n_1:N}(q_2) &< \sum_{n_1=\eta+1}^N p_{n_1:N}(q_2) - \sum_{n_1=\eta+1}^N p_{n_1:N}(q_1) \\ \sum_{n_1=0}^N p_{n_1:N}(q_1) &< \sum_{n_1=0}^N p_{n_1:N}(q_2) \\ 1 &< 1 \end{aligned}$$

which is a contradiction. Therefore, the assumption does not hold and Lemma A2 is proven. \square

Lemma A3. $E[R_{SPA}]$ is continuous and monotonically decreasing in q for a given r .

Proof. Recall that the expected revenue of the FPA is given by

$$E[R_{SPA}] = \sum_{n_2=0}^N p_{n_2:N}(q) R_{SPA}(n_2)$$

where $p_{n_2:N}(q) = \binom{N}{n_2} (1-q)^{n_2} q^{N-n_2}$. For a given risk parameter, r , among N elements of the sum only the expression $(1-q)^{n_2} q^{N-n_2}$ is a function of q . Since it is continuous in q , then the sum of N elements is continuous as well. To show that Π_{SPA} is decreasing in q for a given r , we only need to prove that

$$\sum_{n_2=0}^N [p_{n_2:N}(q_1) - p_{n_2:N}(q_2)] R_{SPA}(n_2) < 0 \quad \text{for } q_1 > q_2$$

To prove by contradiction, assume that

$$\sum_{n_2=0}^N [p_{n_2:N}(q_1) - p_{n_2:N}(q_2)] R_{SPA}(n_2) \geq 0 \quad \text{for } q_1 > q_2$$

From the binomial density function properties we know that $p_{n_2:N}(q_1) > p_{n_2:N}(q_2)$ for small n_2 , and vice versa for large n_2 . Therefore, there exists some η , such that $[p_{n_2:N}(q_1) - p_{n_2:N}(q_2)] R_{SPA}(n_2) \geq 0$ for any $n_2 \leq \eta$, and $[p_{n_2:N}(q_1) - p_{n_2:N}(q_2)] R_{SPA}(n_2) > 0$ for any $n_2 > \eta$. The equation above can be rewritten as follows.

$$\begin{aligned} &\sum_{n_2=0}^{\eta} [p_{n_2:N}(q_1) - p_{n_2:N}(q_2)] R_{SPA}(n_2) \\ &\geq \sum_{n_2=\eta+1}^N [-(p_{n_2:N}(q_1) - p_{n_2:N}(q_2))] R_{SPA}(n_2) \end{aligned}$$

Since $R_{SPA}(n_2)$ is increasing in n_2 , we further have

$$\begin{aligned} &\sum_{n_2=0}^{\eta} [p_{n_2:N}(q_1) - p_{n_2:N}(q_2)] R_{SPA}(n_{\eta+1}) > LHS \\ &\geq RHS \geq \sum_{n_2=\eta+1}^N [-(p_{n_2:N}(q_1) - p_{n_2:N}(q_2))] R_{SPA}(n_{\eta+1}) \end{aligned}$$

This implies the following:

$$\begin{aligned} \sum_{n_2=0}^{\eta} p_{n_2:N}(q_1) - \sum_{n_2=0}^{\eta} p_{n_2:N}(q_2) &> \sum_{n_1=\eta+1}^N p_{n_2:N}(q_2) - \sum_{n_2=\eta+1}^N p_{n_1:N}(q_1) \\ \sum_{n_2=0}^N p_{n_2:N}(q_1) &> \sum_{n_2=0}^N p_{n_2:N}(q_2) \\ 1 &> 1 \end{aligned}$$

which is a contradiction. Therefore, the assumption does not hold and Lemma A3 is proven. \square

B. Example with CRRA bidders

Suppose that bidder i has a utility function of the form $u(m_i) = m_i^{(1-\rho)}$, where m_i represents i 's payoff and $\rho \in [0, 1)$ represents the coefficient of CRRA. Further suppose that values are distributed according to $F(v) = v^\alpha$ for $v \in [0, 1]$, where $\alpha \geq 1$ and takes integer values only. From Smith and Levin (1996), we know that the symmetric equilibrium in FPA is then given by the bidding strategy

$$b^{FPA}(v) = \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} v$$

The ex ante expected revenue of the FPA is given by

$$\begin{aligned} R_{FPA}(n_l, r) &= \int_0^1 n_l \left(\frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} v \right) \alpha v^{\alpha-1} v^{\alpha(n_l-1)} dv \\ &= \alpha n_l \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} \int_0^1 v^{\alpha n_l} dv \\ &= \alpha n_l \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} \left[\frac{1}{\alpha n_l + 1} v^{\alpha n_l + 1} \right]_0^1 \\ &= \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} \frac{\alpha n_l}{\alpha n_l + 1} \end{aligned}$$

Given that there are n_l bidders in the auction, each bidder then has an ex ante expected utility of

$$\begin{aligned} E[u|FPA, n_l] &= \int_0^1 \alpha v^{\alpha-1} v^{\alpha(n_l-1)} \left(v - \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - \rho} v \right)^{1-\rho} dv \\ &= \alpha \left(\frac{1 - \rho}{\alpha(n_l - 1) + 1 - \rho} \right)^{1-\rho} \int_0^1 v^{\alpha n_l - \rho} dv \\ &= \alpha \left(\frac{1 - \rho}{\alpha(n_l - 1) + 1 - \rho} \right)^{1-\rho} \left[\frac{1}{\alpha n_l + 1 - \rho} v^{\alpha n_l + 1 - \rho} \right]_0^1 \\ &= \frac{\alpha}{\alpha n_l + 1 - \rho} \left(\frac{1 - \rho}{\alpha(n_l - 1) + 1 - \rho} \right)^{1-\rho} \end{aligned}$$

For the SPA, the symmetric equilibrium is to bid one's own private value, that is, $b^{SPA}(v) = v$. The ex ante expected revenue of the SPA is then given by

$$\begin{aligned}
R_{SPA}(n_l) &= \int_0^1 n_l(n_l - 1)v\alpha v^{\alpha-1}v^{\alpha(n_l-2)} [1 - v^\alpha] dv \\
&= \alpha n_l(n_l - 1) \int_0^1 v^{\alpha(n_l-1)}(1 - v^\alpha)dv \\
&= \alpha n_l(n_l - 1) \left\{ \left[\frac{1}{\alpha(n_l - 1) + 1} v^{\alpha(n_l-1)+1} \right]_0^1 - \left[\frac{1}{\alpha n_l + 1} v^{\alpha n_l + 1} \right]_0^1 \right\} \\
&= \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1} \frac{\alpha n_l}{\alpha n_l + 1}
\end{aligned}$$

Following Smith and Levin (1996), we show that, given that there are n_l bidders in the auction, each bidder has an ex ante expected utility of

$$\begin{aligned}
E[u|SPA, n_l] &= \int_0^1 \left[\alpha(n_l - 1) \int_0^v t^{\alpha(n_l-1)-1} (v - t)^{1-\rho} dt \right] \alpha v^{\alpha-1} dv \\
&= \frac{\alpha}{\alpha n_l + 1 - \rho} \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!}
\end{aligned} \tag{B1}$$

where $(\alpha(n_l - 1) + 1 - \rho)! \equiv \prod_{i=1}^{\alpha(n_l-1)} (i + 1 - \rho)$. To establish (B1), we start by proving that the term in square brackets in (B1), which represents the expected utility of a bidder with value v , can be rewritten as follows.

$$\alpha(n_l - 1) \int_0^v t^{\alpha(n_l-1)-1} (v - t)^{1-\rho} dt = \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!} v^{\alpha(n_l-1)+1-\rho} \tag{B2}$$

Suppose that $\alpha(n_l - 1) = 1$. Then (B2) is trivially true.

$$\int_0^v (v - t)^{1-\rho} dt = -\frac{1}{2 - \rho} \left[t^{2-\rho} \right]_0^v = \frac{1}{2 - \rho} v^{2-\rho}$$

Let us now show that (B2) also holds for $\alpha(n_l - 1) = 2$. In order to do so, we need to use integration by parts: $\int u dv = uv - \int v du$. Integrating the LHS of (B2) by parts gives us the following.

$$\begin{aligned}
&-\alpha(n_l - 1) \left[\frac{1}{2 - \rho} t^{\alpha(n_l-1)-1} (v - t)^{2-\rho} \right]_0^v \\
&+ \alpha(n_l - 1) \frac{\alpha(n_l - 1) - 1}{2 - \rho} \int_0^v t^{\alpha(n_l-1)-2} (v - t)^{2-\rho} dt \\
&= \frac{\alpha(n_l - 1)(\alpha(n_l - 1) - 1)}{2 - \rho} \int_0^v t^{\alpha(n_l-1)-2} (v - t)^{2-\rho} dt
\end{aligned} \tag{B3}$$

Now suppose that $\alpha(n_l - 1) = 2$. The RHS of (B3) then becomes

$$\begin{aligned} & \frac{\alpha(n_l - 1)(\alpha(n_l - 1) - 1)}{2 - \rho} \int_0^v (v - t)^{2-\rho} dt \\ &= -\frac{\alpha(n_l - 1)(\alpha(n_l - 1) - 1)}{2 - \rho} \left[\frac{1}{3 - \rho} (v - t)^{3-\rho} \right]_0^v \\ &= \frac{2 * 1}{(2 - \rho)(3 - \rho)} v^{3-\rho} \end{aligned}$$

which proves that (B2) holds for $\alpha(n_l - 1) = 2$ as well. Having verified (B2) for $\alpha(n_l - 1) = \{1, 2\}$ we now prove by induction. Assume that (B2) holds for $\alpha(n_l - 1) = k$.

$$k \int_0^v t^{k-1} (v - t)^{1-\rho} dt = \frac{k!}{(k + 1 - \rho)!} v^{k+1-\rho} \quad (\text{B4})$$

Now, we can show that (B2) also holds for $\alpha(n_l - 1) = k + 1$. That is, we want to prove the following.

$$(k + 1) \int_0^v t^k (v - t)^{1-\rho} dt = \frac{(k + 1)!}{((k + 1) + 1 - \rho)!} v^{(k+1)+1-\rho} \quad (\text{B5})$$

We start by integrating the LHS of (B5). This gives us the following.

$$\begin{aligned} & (k + 1) \left\{ - \left[\frac{1}{2 - \rho} t^k (v - t)^{2-\rho} \right]_0^v + \frac{k}{2 - \rho} \int_0^v t^{k-1} (v - t)^{2-\rho} dt \right\} \\ &= \frac{(k + 1)}{(2 - \rho)} \left\{ k \int_0^v t^{k-1} (v - t)^{2-\rho} dt \right\} \end{aligned}$$

We now use B4 to rewrite this as follows.

$$\frac{(k + 1)}{(2 - \rho)} \left\{ \frac{k!}{(k + 2 - \rho)!} v^{k+2-\rho} \right\} = \frac{(k + 1)!}{(k + 2 - \rho)!} v^{k+2-\rho}$$

This establishes (B5) and concludes the proof of (B2). Therefore, we can write the ex ante expected utility, where the bidder does not know her private value yet, as follows.

$$\begin{aligned} E[u|SPA, n_l] &= \int_0^1 \left[\frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!} v^{\alpha(n_l - 1) + 1 - \rho} \right] \alpha v^{\alpha - 1} dv \\ &= \alpha \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!} \int_0^1 v^{\alpha n_l - \rho} dv \\ &= \alpha \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!} \left[\frac{1}{\alpha n_l + 1 - \rho} v^{\alpha n_l + 1 - \rho} \right]_0^1 \\ &= \frac{\alpha}{\alpha n_l + 1 - \rho} \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - \rho)!} \end{aligned} \quad (\text{B6})$$

This concludes the proof of (B1).

Notice that when bidders are risk averse ($\rho = 0$), then the FPA and SPA are both revenue and utility equivalent.

$$\begin{aligned} R_{FPA}(n_l, 0) &= \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1 - 0} \frac{\alpha n_l}{\alpha n_l + 1} \\ &= \frac{\alpha(n_l - 1)}{\alpha(n_l - 1) + 1} \frac{\alpha n_l}{\alpha n_l + 1} = R_{SPA}(n_l) \end{aligned}$$

$$\begin{aligned}
E[u|FPA, n_l] &= \frac{\alpha}{\alpha n_l + 1 - 0} \left(\frac{1 - 0}{\alpha(n_l - 1) + 1 - 0} \right)^{1-0} \\
&= \frac{(\alpha(n_l - 1))!}{(\alpha(n_l - 1) + 1 - 0)!} \frac{\alpha}{\alpha n_l + 1 - 0} = E[u|SPA, n_l]
\end{aligned}$$