The life and work of Caucher Birkar, the migrant mathematician

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Nationale Wiskunde Dagen,
February 1, 2020

Maths makes the news, 1 August 2018

International Congress of Mathematichians, Rio de Janerio.

Maths makes the news, 2 August 2018

The New York Times

Fields Medal Is Stolen Minutes After It's Given in Brazil

Caucher Birkar, a professor at the University of Cambridge in England, receiving the Fields Medal in Rio de Janeiro on Wednesday. Fabio Teixeira/Agence France-Presse —

Official announcement

Short Bio of Caucher Birkar

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- 1979: Iranian Revolution.
- 1980-1988: Iraq-Iran war.

Short bio of Cauher Birkar

"My brother Haidar started teaching me more advanced mathematics, which was not part of middle school education. He taught me to learn maths for the fun of it rather than passing exams.
He is probably the person that had the most influence on my education."

Birkar's family.

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- 2002: He meets Vyacheslav Shokurov and starts working in birational geometry.
- 2010: He proves a major result in collaboration with Cascini, Hacon and McKernan [BCHM], he becomes professor at Cambridge and a maths superstar.

Birkar is an Algebraic Geometer

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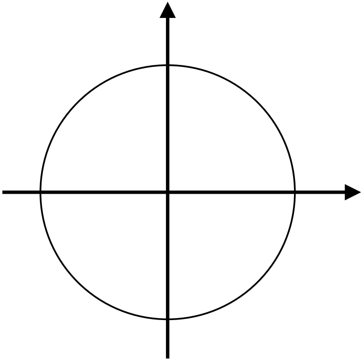
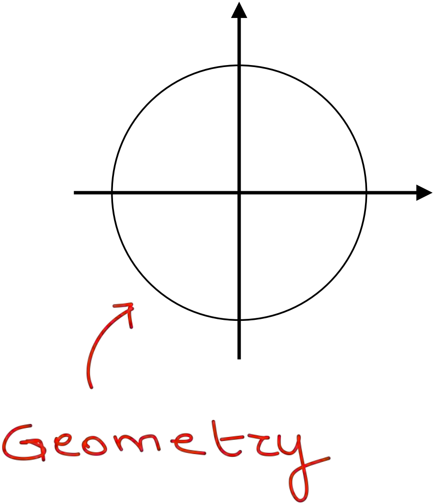
- Geometer $\Rightarrow$ He thinks about geometric objects.
- Algebraic $\Rightarrow$ He uses algebraic tools.

The Circle



$$x^2 + y^2 = 1$$

The Circle



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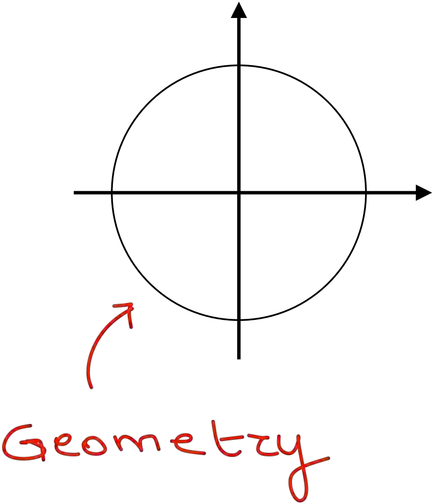


$$x^2 + y^2 = 1$$

Algebra

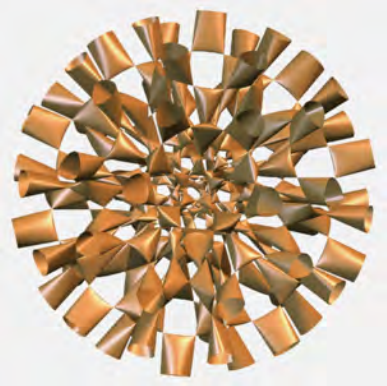
Polynomials are everywhere

- Simple functions.
- They model many naturally-occurring processes across science.
- They approximate smooth functions.

Sydney Harbour Bridge

We call the space of solutions of polynomial equations **algebraic varieties**.

The zoo of polynomials



The dodecic surface. It was discovered by Alessandra Sarti in 2008.

$X_{12} = 243S_{12} - 22Q_{12} = 0$

where

$Q_{12} = (x^2 + y^2 + z^2 + w^2)^6$

$S_{12} = 33\sqrt(5)(s^-_{2,3} + s^-_{3,4} + s^-_{4,2}) + 19(s^+_{2,3} + s_{3,4} +^+ + s^+_{4,2})$
$+ 10s_{2,3,4} - 14s_{1,0} + 2s_{1,1} - 6s_{1,2}$
$- 352s_{5,1} + 336l_5^2 l_1 + 48l_2 l_3 l_4$

$$l_1 = x^4 + y^4 + z^4 + w^4$$

$$l_2 = x^2y^2 + z^2w^2$$

$$l_3 = x^2z^2 + y^2w^2$$

$$l_4 = x^2w^2 + y^2z^2$$

$$l_5 = xyzw \ldots$$

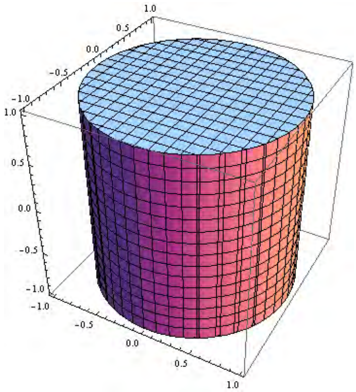
Classification

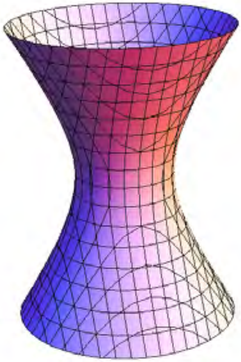
Caucher Birkar:

"We want to bring order to an infinitude of equations."

Comparing varieties I

Cylinder vs Hyperboloid





Isomorphism

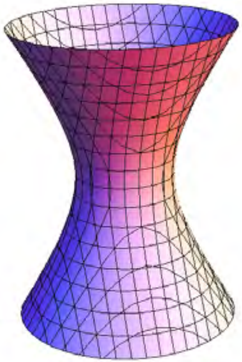
We say that two varieties are **isomorphic** if there are morphisms $f : X \to Y$ and $h : Y \to X$ that are inverse of each other.

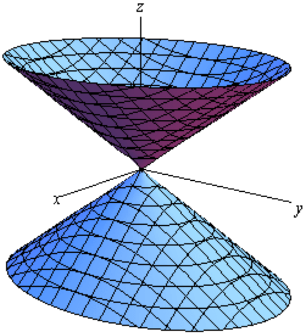
Example:

The cylinder and the hyperboloid are isomorphic.

Comparing varieties II

Hyperboloid vs Cone





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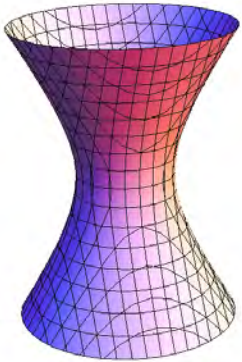

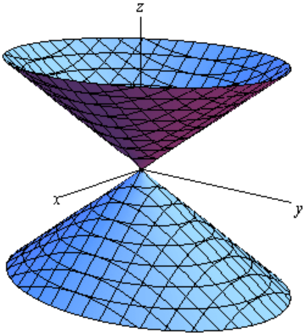

The hyperboloid and the cone are **NOT** isomorphic.

Singular vs Smooth

Going from the hyperboloid to the cone we created a **singularity**, i.e. a point where all the partial derivatives vanish.

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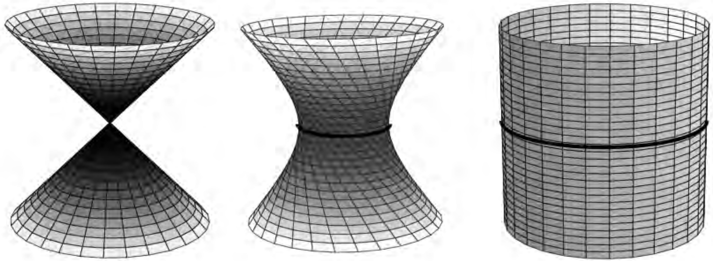
Singularities are **evil**!

We want them all into the same class!



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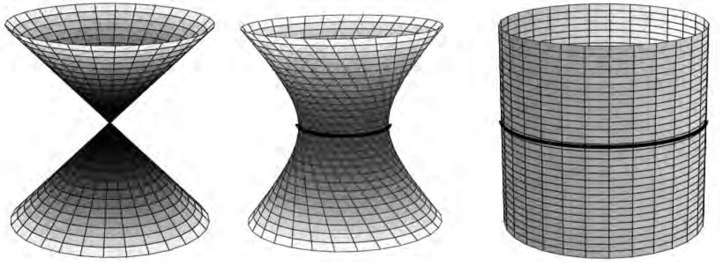

We want to allow **small modifications**.

Birational Equivalence

Definition

X and Y are **birationally equivalent** if there are closed subsets $Z \subset X$ and $W \subset Y$ such that $X \backslash Z$ is isomorphic to $Y \backslash W$.

Example

The hyperboloid and the cone are birationally equivalent.

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The hyperboloid and the cone are birationally equivalent.

- From the cylinder to the cone: the black loop collapses into the singularity.
- From the cone to the cylinder: we are resolving the singularity adding the black loop that is parametrizing all the colliding directions. This is what we call **a blow-up.**

The birational classification

Hironaka's resolution of singularities, 1964

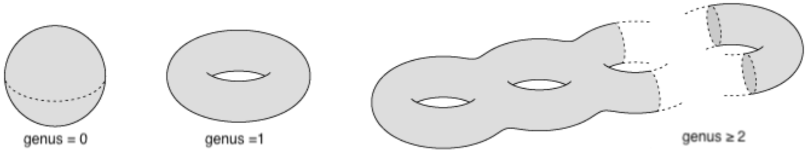
Over $\mathbb{C}$, every algebraic variety is birationally equivalent to a smooth variety obtained via a series of blow-ups.

Our goal

We want to classify smooth projective varieties defined over the complex numbers **up to birational equivalence**.

What are we hoping to obtain?

For smooth projective **curves** we have a satisfactory classification.



deg(f)	g(X)	Variety	Universal cover	Curvature
1,2	0	$\mathbb{P}^1$	S^2	> 0
3	1	Elliptic curves	$\mathbb{R}^2$	$= 0$
> 3	≥ 2	Hyperbolic curves	$\lvert x\rvert \leq 1 \subset \mathbb{R}^2$	< 0

Higher dimension?

- As soon as the dimension of the variety is greater than one, there are too many smooth varieties in the birational equivalence class.
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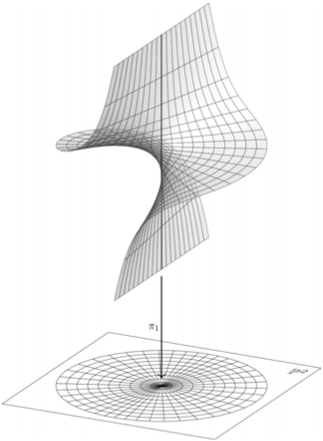
We are looking for a representative that is **minimal**.

Birational surfaces I



All the time that we blow up a point on a smooth surface, we produce a special curve E. We call E an **exceptional curve**.

Minimal surfaces

Definition

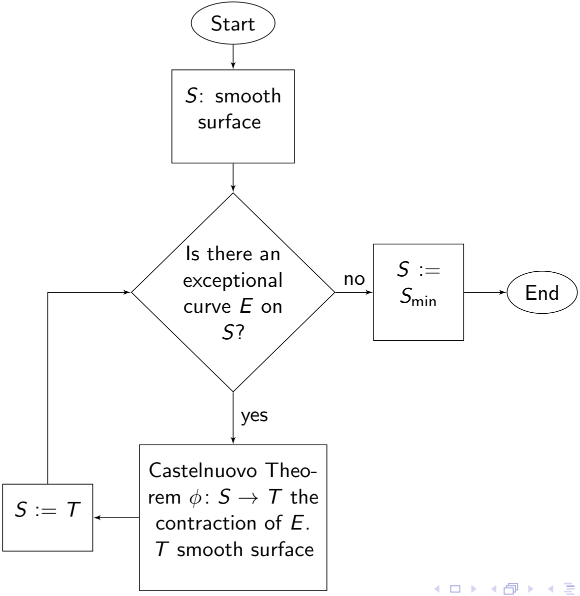
A smooth surface is **minimal** if it does not contain any exceptional curves.

Definition

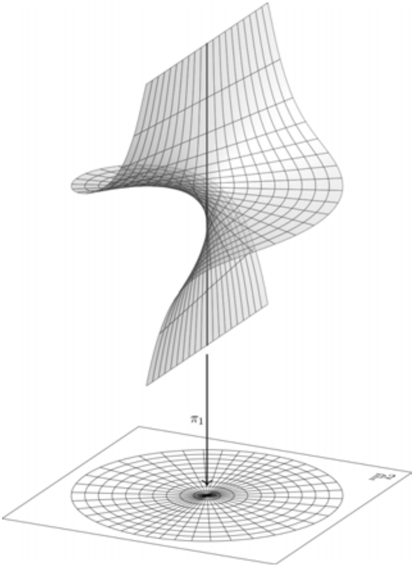
Let X be a smooth surface. A minimal surface Y birational to X is called a **minimal model for** X.

How do we find the minimal model inside the birational equivalence class?

Minimal Model Program in dimension 2



Birational surfaces II



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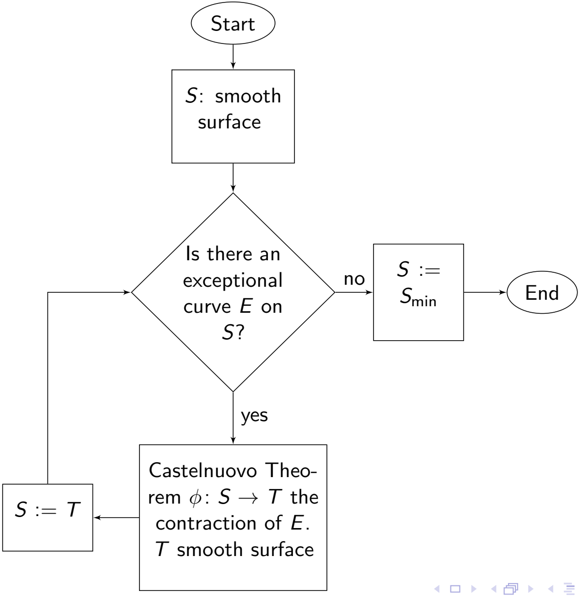

Classification of surfaces

In a finite number of steps we can reach the minimal surface in the birational equivalence class.

Minimal surfaces are classified.

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Minimal surfaces are classified.
Greatest achievement of the Italian School of algebraic geometry.

MMP: state of the art

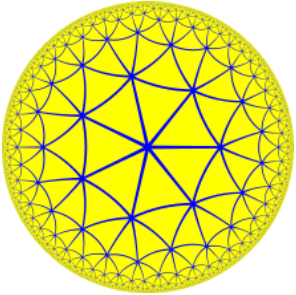
The program has been completely solved in dimension 3 [Mori '80s, . . .], 4 [Shokurov '90s, . . .], 5 [Birkar2009], and for an important class of varieties called of general type [**B**irkar, **C**ascini, **H**acon, **M**^{c}Kernan, 2010].

Mori, Fields medal in 1990.

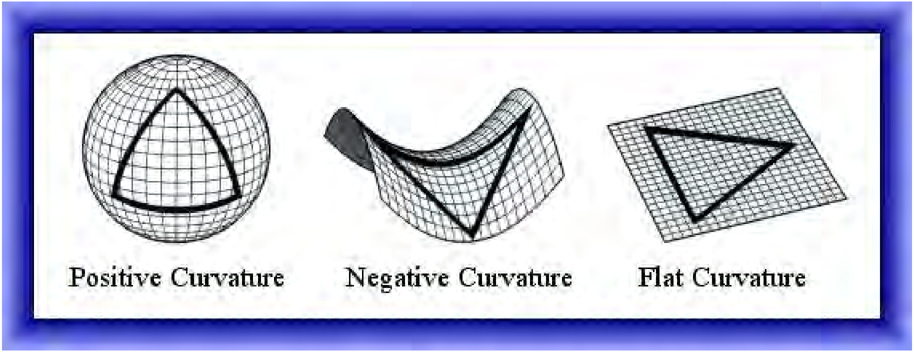
McKernan and Hacon, Breakthrough prize in 2017.

Birkar, Fields medal in 2018.

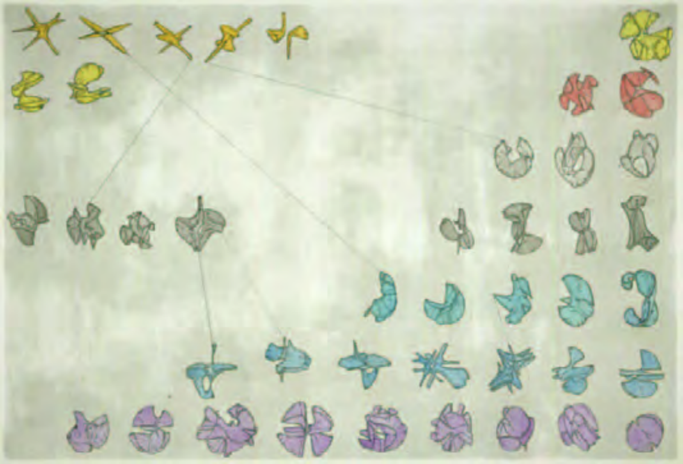
The building blocks

Fano varieties	Canonically polarized	Calabi-Yau varieties
		
Courtesy of The FanoSearch Project.	Hurwitz surface.	Kummer surface.

Trichotomy in terms of curvature



The periodic table of shapes



Watercolour by London artist Gemma Anderson.

Official announcement

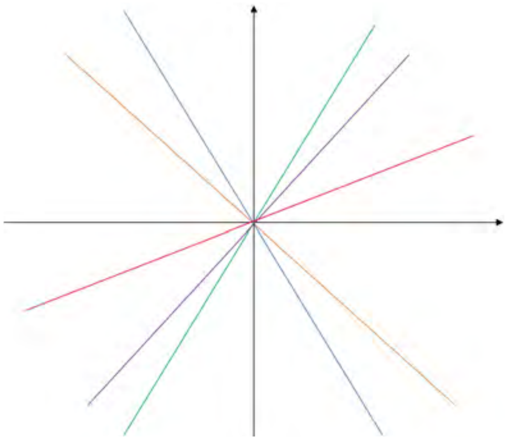
Boundedness of Fano varieties

In 2016, Birkar proved the **BAB (Borisov-Alexeev-Borisov) conjecture** that states that Fano varietes with a special class of singularities form a *bounded family*.

What does being a *bounded family* mean?

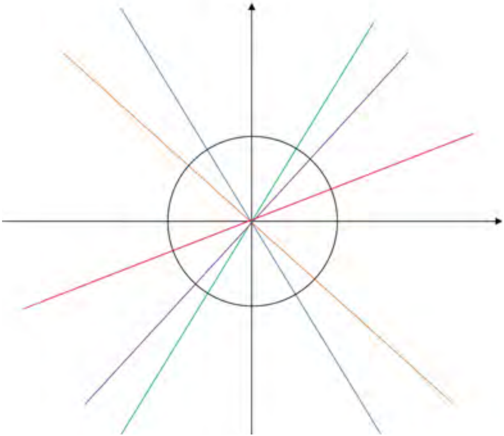
Bounded means you can describe all the (infinitely many) shapes in the family with a **finite** number of parameters.

Geometric idea



Lines passing through the origin

Geometric idea



Lines passing through the origin are parametrized by a **circle**.

In terms of algebra

We can describe the elements of the family with a **finite set of variables** and equations of a **bounded** degree.

Different elements in the family corresponds to different choices of coefficients:

$$5y + 3x = 0 \qquad 2y + \frac{7}{9}x = 0.$$

Thank you!