CLUexercise 1, Bifurcations of the wind-driven ocean circulation
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1 Background

The theory of the homogeneous wind-driven ocean circulation is one of the cornerstones in modern physical oceanography. Consider a rectangular ocean basin of size $L \times L$ having a constant depth D. The basin is situated on a midlatitude β-plane with a central latitude $\theta_0 = 45°$N and Coriolis parameter $f_0 = 2\Omega \sin\theta_0$, where Ω is the rotation rate of the Earth. The meridional variation of the Coriolis parameter at the latitude θ_0 is indicated by β_0. The density ρ of the water is constant and the flow is forced at the surface through a wind-stress vector $\mathbf{T} = \tau_0[\tau^x(x,y), \tau^y(x,y)]$. The governing equations are non-dimensionalized using a horizontal length scale L, a vertical length scale D, a horizontal velocity scale U, the advective time scale L/U and a characteristic amplitude of the wind-stress vector, τ_0. The effect of deformations of the ocean-atmosphere interface on the flow is neglected.

The dimensionless barotropic quasi-geostrophic model of the flow for the vorticity ζ and the geostrophic streamfunction ψ is

$$\Big[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\Big][\zeta + \beta y] = Re^{-1}\nabla^2\zeta - \mu\zeta + \alpha_\tau\Big(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\Big), \quad (1a)$$

$$\zeta = \nabla^2\psi, \quad (1b)$$

where the horizontal velocities are given by $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$. The parameters in (1) are the Reynolds number Re, the planetary vorticity gradient parameter β and the wind-stress forcing strength α_τ. These parameters are defined as:

$$Re = \frac{UL}{A_H}; \quad \alpha = \frac{\tau_0 L}{\rho D U^2}; \quad \beta = \frac{\beta_0 L^2}{U}; \quad F = \frac{f_0^2 L^2}{gD}; \quad \mu = \frac{L\epsilon_0}{U}. \quad (2)$$

where A_H and ϵ_0 are the lateral and bottom friction coefficients, respectively. When the horizontal velocity scale is based on a Sverdrup balance of the flow, i.e.,

$$U = \frac{\tau_0}{\rho D \beta_0 L}. \quad (3)$$

it follows that $\alpha_\tau = \beta$ and three free parameters result.

We assume no-slip conditions on the east-west boundaries and slip on the north-south boundaries. The boundary conditions are therefore given by

$$x = 0, x = 1 \quad : \quad \psi = \frac{\partial\psi}{\partial x} = 0, \quad (4a)$$

$$y = 0, y = 1 \quad : \quad \psi = \zeta = 0. \quad (4b)$$

The wind-stress forcing is prescribed as

$$\tau^x(x,y) = \frac{-1}{2\pi}\cos 2\pi y, \quad (5a)$$

$$\tau^y(x,y) = 0, \quad (5b)$$

and the zonal wind stress is symmetric with respect to the mid-axis of the basin (the standard double-gyre case).

The classical Sverdrup-Munk theory of the homogeneous wind-driven ocean circulation is linear. Note that on the basin scale L, a balance exists between the advection of planetary vorticity (β-effect) and the wind-stress curl. In the boundary layers near the meridional walls, a balance between friction and the β-effect occurs; this can only give substantial flow in the western boundary layer. If nonlinear terms are included, unexpected phenomena can happen. Although this can in principle be analysed in the model above, we will investigate a few of these phenomena in a severe truncation of this model.

Background literature: Nonlinear Physical Oceanography, Chapter 5, H. A. Dijkstra, 2005.

2 Mathematical problem

The low-order model derivation starts from the barotropic vorticity equation by taking zero lateral mixing ($Re \to \infty$) and keeping non-zero bottom-friction. On the boundaries only no normal flow conditions, i.e., $\psi = 0$ are applied. Next step is to project the equations using suitable expansion functions. In order to account for the existence of the western boundary layer, a decaying exponential in the x direction is introduced while a sine expansion is retained in the y direction, i.e.

$$\begin{aligned}
\psi &= A_1(t)G(x)\sin y + A_2(t)G(x)\sin 2y + \\
&+ A_3(t)G(x)\sin 3y + A_4(t)G(x)\sin 4y \\
G(x) &= e^{-sx}\sin x
\end{aligned}$$

where $s = 1.3$ is chosen such that $G(x)$ fits with the zonal asymmetric structure of a typical flow. The truncated equations are obtained by projecting the barotropic voriticity equation onto the orthogonal basis ($G(x)\sin ky$, k=1,4) using the inner product $< f, g >= \int_0^\pi \int_0^\pi fg\; dxdy$ such that the energy of the truncated system is conserved.

The amplitudes of the modes are indicated by $A_i, i = 1, .., 4$ and the model contains one control parameter σ. All the other parameters are fixed. The model is written as the set of ODEs

$$\frac{dA_1}{dt} = c_1 A_1 A_2 + c_2 A_2 A_3 + c_3 A_3 A_4 - l_1 A_1 \tag{7a}$$

$$\frac{dA_2}{dt} = c_4 A_2 A_4 + c_5 A_1 A_3 - c_1 A_1^2 - l_2 A_2 + c_7 \sigma \tag{7b}$$

$$\frac{dA_3}{dt} = c_6 A_1 A_4 - (c_2 + c_5) A_1 A_2 - l_3 A_3 \tag{7c}$$

$$\frac{dA_4}{dt} = -c_4 A_2^2 - (c_3 + c_6) A_1 A_3 - l_4 A_4 \tag{7d}$$

with $c_1 = 0.020736, c_2 = 0.018337, c_3 = 0.015617, c_4 = 0.03197, c_5 = 0.036673, c_6 = 0.046850, c_7 = 0.314802, l_1 = 0.0128616, l_2 = 0.0211107, l_3 = 0.0318615, l_4 = 0.0427787$.

3 Numerical problem

In this exercise, we will determine fixed points, periodic orbits of the system above, using the package AUTO.

Login to clue.science.uu.nl and go to the directory of 4mode. Follow the directions to run the code and post process the output in the README file.

a) Start at the zero solution for $\sigma = 0$ and determine a branch of fixed points in σ. Determine the σ value of the first pitchfork bifurcation.

b) What is the internal symmetry of the system giving rise to the pitchfork bifurcation? Describe the physical mechanism of the symmetry breaking.

c) Determine the branches of asymmetric solutions. As the bifurcation diagram, plot the value of A_1 versus σ.

d) Determine the value of σ at the Hopf bifurcation on one of the branches of asymmetric solutions.

e) Calculate a branch of periodic solutions from the Hopf bifurcation. Are these periodic orbits stable?

f) Follow the periodic orbits up to very large period and find the approximate value of σ where the homoclinic connection occurs.