The geometry of the Game DOBBLE

The goal of this handout is to expose the geometry behind the game Dobble. In particular, we present how Mathematics can be used to construct the game and other versions with different numbers of symbols and cards.

It is aimed at (Middelbare) school teachers.

The game

The game comes with a deck of 55 cards, each having 8 symbols and such that any 2 given cards share precisely 1 symbol.

There are many ways to play the game, but one common goal, namely, to be the first one to spot the common symbol in as many pairs of cards as possible.

Example of a Dobble game: One of the ways starts with each player receiving a card facing down, one open card in the middle, and the remaining cards facing down in a pile. The players flip their cards up and the first one to name the common symbol between their card and the middle one takes the middle card. A new card from the pile is revealed and the game continues until there are no more cards left in the pile. The winner is the player with the most cards.

History

The history of Dobble goes back to the 19th century. In 1850, Reverend Thomas Penyngton Kirkman submitted the following puzzle to the *Ladies and Gentleman Diary,* an annual recreational mathematical magazine read by both amateurs and professional mathematicians. *"Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."* This combinatorial problem was solved by Kirkman himself a few years before. It fits naturally as a special case of a general problem that asks in how many ways can we combine n symbols in groups of p symbols such that no combination of q symbols that appear in one combination can be repeated in another (in Kirkman $(n, p, q) = (15,3,2)$).

A solution to the general problem only appeared in 1968 and, with it, an increased interest in Kirkman's schoolgirl problem. In 1976, Jacques Cottereau, a French math enthusiast created the *Game of insects,* inspired by Kirkman's problem. It had a deck of 31 cards with 6 images of insects in each so that each card had exactly one symbol in common.

Cotterau's game was never commercialized. About 30 years later, in 2008, Denis Blanchot, a journalist rediscovered the Game of insects and adapted it, creating Dobble.

Building our own game

As an exercise, we start with a small version of Dobble.
Build a game using 7 different symbols to be distributed among 7 cards with 3 symbols per card (21=3x7 symbols appear in the whole deck) such that any 2 given cards share precisely 1 symbol.

How does it work in general?

We would like to answer the following questions:

1) How many pictures are there in the whole deck?
2) How many times does a picture appear in the whole deck?
3) Can we play the game if we remove a card?

We would also like to be able to rebuild our own game of Dobble.

Geometry (and a bit of algebra)

In what follows we introduce the concept of point-line geometry, which allows us to interpret the set-up of the game in a mathematical language.

I) Point-line geometry

For the sake of simplicity, we restrict ourselves to finite geometries, i.e., geometries with a finite set of points.

A point-line geometry consists of a set $S = \{p_1, \ldots, p_n\}$, whose elements we call points, and a family of subsets $\{L_1, \ldots, L_m\}$, called the lines, satisfying the following axioms:

A1: Two points are in at most one line;
A2: Two lines meet in at most one point;
A3: A line contains at least 2 points;
A4: There are 4 points such that no 3 of them are on a line.

We apply this to our game

DOBBLE	GEOMETRY
Symbol	Point
Card	Line

DOBBLE	GEOMETRY
Any two cards have one symbol in common.	Any two lines meet in one point.
Any two symbols belong to exactly one card.	Any two points determine a line.

The last two statements in the table show that points and lines in this geometry satisfy a stronger property than stated in the axioms above:

Any two lines meet. Hence there are no parallel lines!

The geometry of Dobble is a special instance of the so-called projective plane over $\mathbb{F}_p$, where p is a prime number.

In what follows we introduce enough theory to allow you to understand the previous sentence and come up with your own version of Dobble with $p+1$ symbols per card for any prime number p.

Modular arithmetic

The integers modulo n is the set $\{0,\ldots,n-1\}$ together with the usual addition, subtraction, and multiplication subject to the following rule: whenever a sum $a+b$, a difference $a-b$, or a product $a \cdot b$ is outside the set $\{0,\ldots,n-1\}$, we add or subtract a multiple of n so that the result lies in the set.

Notation: If $a+b=k\cdot n+r$, with $r\in\{0,1,\ldots,n-1\}$ we write $a+b\equiv r \mod n$.

Example a) Integers modulo 6

$$3+4=7=1+6\equiv 1 \mod 6$$
$$3-4=-1=5-6\equiv 5 \mod 6$$
$$3\cdot 4=12=0+2\cdot 6\equiv 0 \mod 6$$

Example b) Integers modulo 5

$$3+4=7=5+2\equiv 2 \mod 5$$
$$3-4=-1=4-5\equiv 4 \mod 5$$
$$3\cdot 4=12=10+2\equiv 2 \mod 5$$

Addition and Multiplication tables modulo 5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Division

On the other hand, the division might cause problems.

For instance, in the integers modulo 6 as in example a) we see that

$$3 \cdot 3 \equiv 9 \equiv 3 \cdot 1 \mod 6.$$

So it is not clear if $3 \div 3$ is 1 or 3.
The reason for this inconsistency is that 3 divides 6.

Restricting ourselves to prime numbers circumvents this problem and allows us to divide.

In example b) for instance, $12 \equiv 2 \mod 5$. Division by 2 gives $6 \equiv 1 \mod 5$.

Remark: Given a prime number p, the fact that we can add, subtract, multiply and divide allows us to write down equations for lines through two given points.

Notation: Given a prime p, we denote the set of integers modulo p together with the addition, multiplication, subtraction, and division as above by $\mathbb{F}_p$.

We introduce a geometry that allows us to interpret the game Dobble.

The projective plane over $\mathbb{F}_p$

We consider a point-line geometry whose points are of the form $(a, b, c) \in \mathbb{F}_p^3 \backslash \{(0,0,0)\}$ subject to $(\lambda a, \lambda b, \lambda c) = (a, b, c), \forall \lambda \in \mathbb{F}_p \backslash \{0\}$.
The lines are given by equations $\alpha x + \beta y + \gamma z = 0; (\alpha, \beta, \gamma) \in \mathbb{F}_p^3 \backslash \{(0,0,0)\}$.
We denote a point $(a, b, c) = (\lambda a, \lambda b, \lambda c) \in S$ by $(a : b : c)$.

The four operations in $\mathbb{F}_p$ guarantee that the following properties hold:

P1. Any line contains at least two points.
P2. Two points are on precisely one line.
P3. Two lines intersect at precisely one point.
P4. There exists a set of four points, no three of which are collinear.

Point x line: We say that a point $(a : b : c)$ is on the line $\alpha x + \beta y + \gamma z$ if it satisfies the equation $\alpha x + \beta y + \gamma z = 0$, i.e., $\alpha a + \beta b + \gamma c = 0$.

Duality: There are as many points as lines in this geometry. Indeed, to a point, $(a : b : c) \in S$, we can associate a line $ax + by + cz$. Vice versa, to a line $\alpha x + \beta y + \gamma z$ we can associate the point $(\alpha : \beta : \gamma)$.

Notation: We denote the set of points in the projective plane over $\mathbb{F}_p$ by $\mathbb{P}^2(\mathbb{F}_p)$.

In what follows, we count points, lines, and points in a line in the projective plane over $\mathbb{F}_p$.

Proposition 1
The projective plane over $\mathbb{F}_p$ has $p^2 + p + 1$ points and $p^2 + p + 1$ lines.

Proposition 2
Each line has $p + 1$ points and each point is contained in $p + 1$ lines.

This allows us to revisit the mini-version of Dobble with 7 cards.

The mathematics of mini-Dobble

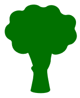
"Each card has 3 symbols" corresponds to "each line has 3 points". Proposition 2 implies that $p = 2$. Hence we work on the projective plane over $\mathbb{F}_2$.

Symbols can be represented as
$(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 1), (1 : 0 : 1), (0 : 1 : 1), (1 : 1 : 0)$.

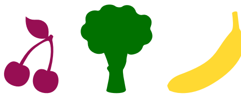
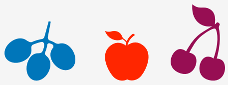
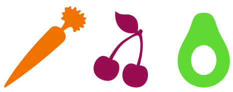
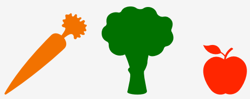
How do we construct our game?

Any two symbols (points) belong to a unique card (line). Using the equation of the line we discover the third symbol on that card.

Example: $(0 : 0 : 1)$ and $(0 : 1 : 0)$ belong to the line $x = 0$.
Hence the third point in the card containing $(0 : 0 : 1)$ and $(0 : 1 : 0)$ is $(0 : 1 : 1)$.
We can associate each point with a symbol

Point	Symbol	Point	Symbol
$(0:0:1)$		$(1:0:1)$	
$(0:1:0)$		$(1:1:0)$	
$(1:0:0)$		$(1:1:1)$	
$(0:1:1)$			

Looking at which 3 symbols are collinear, we can form our cards.

Line	Card	Line	Card
$x = 0$		$x + z = 0$	
$y = 0$		$y + z = 0$	
$z = 0$		$x + y + z = 0$	
$x + y = 0$			

Answering the questions:

1) There are $2^2 + 2 + 1 = 7$ pictures in the game and 7 cards on the whole deck.
2) Any given picture appears $2 + 1 = 3$ times in the whole deck.
3) Yes. Removing a card is equivalent to removing a line from the projective plane. Any two lines among the remaining ones still intersect at a unique point.

Exercise: Can you recreate Cotterau's game of insects?

How to build Dobble using geometry?

In Dobble, each card has $8 = 7 + 1$ symbols. Hence we work with the projective plane over $\mathbb{F}_7$.
The whole deck has $7^2 + 7 + 1 = 57$ symbols.

The 57 symbols can be thought of as elements of $\mathbb{P}^2(\mathbb{F}_7)$. These are

$$(0:0:1), (0:1:0), (1:0:0), (1:1:0), (1:0:1), (0:1:1), \ldots, (1:6:5)$$

For any two given symbols, there is a unique card containing them.
But, how do we know that 3 or more given symbols belong to a card?

Check whether they are in the same line!

To form our deck, we can start by taking pairs of points, for instance, $(0:0:1)$ and $(0:1:0)$, and look for all the points that lie on the line determined by them. In this case, the line is $x = 0$ and the points are

$$(0:1:1), (0:1:2), (0:1:3), (0:1:4), (0:1:4), (0:1:6).$$

Alternatively, we can write down the equations for the 57 lines in $\mathbb{P}^2$ over $\mathbb{F}_7$ and check which points lie in each of the lines.

Curiosity: The game Dobble has 55 cards, and not 57. This is not a problem since the game can be played if we remove cards from the deck. Some believe that this is due to fabrication constraints: a standard card deck has in general 52 playing cards, 2 jokers and an advertising card giving a total of 55 cards.