On the Embedding of Manifolds into the Smooth Zariski Topos

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In [NR3], Reyes and the author described a model for "synthetic differential geometry" (see [K]) in which both nilpotent infinitesimals and invertible infinitesimals are present. The first type of infinitesimals occur in the work of geometers like G. Darboux, S. Lie, and E. Cartan, who often used "synthetic" reasoning. The second type of infinitesimals occur in the work of many physicists, for example, in the discussion of the $\theta$-function of Dirac. These invertible infinitesimals have been extensively studied in nonstandard analysis (NSA, see [R]), but NSA is inconsistent with the existence of nilpotent infinitesimals as used in synthetic differential geometry, as pointed out in [K].

The model from [NR3] mentioned above is an analogue of the Zariski topos, the so-called smooth Zariski topos, denoted by $\mathcal{Z}$. $\mathcal{Z}$ is an extension of the category of smooth manifolds $\mathcal{M}$, i.e., there is a full embedding $s : \mathcal{M} \rightarrow \mathcal{Z}$.

In addition, however, $\mathcal{Z}$ contains infinitesimal spaces, and -- as in any Grothendieck topos -- arbitrary function spaces can be formed.

One of the main points made in [NR3] was that it is the object $\mathbb{N} = \text{the image of } \mathbb{N}$ of the discrete manifold of natural numbers, and not the natural number object of $\mathcal{Z}$, which has to be used to do analysis and algebra in $\mathcal{Z}$. (Or from a more logical point of view, one may say that the logic and arithmetic of $\mathcal{Z}$ are not intuitionistic higher order logic and full Heyting arithmetic, but intuitionistic higher order logic and a weaker system for arithmetic, with a restricted induction axiom, see [NR3], [MR4].) The aim of this paper is to show that the embedding $s : \mathcal{M} \rightarrow \mathcal{Z}$ preserves the usual topological properties of manifolds, provided these properties are formulated in the correct way, again using the object $\mathbb{N} = \text{the image of } \mathbb{N}$, rather than the natural number object of the topos. I will concentrate on properties relevant to the development of homology and De Rham cohomology in smooth toposes like $\mathcal{Z}$ (cf. [MR2]), such as compactness, connectedness, partitions of unity, and open covers. The proofs of these preservation properties of the embedding $s : \mathcal{M} \rightarrow \mathcal{Z}$ reveal a very interesting, and -- at least for me -- rather unexpected relation between the study of models of synthetic differential geometry and dimension theory, given by the crucial use that is made of Ostrand's covering characterization of dimension.

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1. The Zariski topos

In this preliminary section, we will describe the smooth Zariski topos, and explain how and why it is a model of synthetic differential geometry. Much of the material of this section can also be found in [HR3]. The main novelties are the axiom of bounded search (see 1.5 below), and the fact that the definition of the smooth Zariski topos given here differs slightly from the one given in [HR3].

1.1. The category \( \mathbf{L} \) of loci. This is simply the dual of the category of finitely generated \( \mathbb{C} \)-rings. Explicitly, the objects of \( \mathbf{L} \) are formal duals \( \overline{A} \) of \( \mathbb{C} \)-rings (isomorphic to those of the form

\[
A = \mathbb{C}(\mathbb{R}^n)/I,
\]

where \( \mathbb{C}(\mathbb{R}^n) \) is the ring of smooth functions \( \mathbb{R}^n \to \mathbb{R} \), and \( I \) is any ideal. The maps of \( \mathbf{L} \) from one such dual \( \mathbb{C}(\mathbb{R}^n)/I \) to another \( \mathbb{C}(\mathbb{R}^m)/J \) are equivalence classes of smooth functions \( \mathbb{R}^n \to \mathbb{R}^m \) with the property that \( f \in J = f + \phi \in I \), two such functions \( \phi \) and \( \psi \) being equivalent if \( \phi - \psi \in I \) (for \( i = 1, \ldots, m \)). Up to isomorphism, the objects of \( \mathbf{L} \) include duals of rings of the form \( \mathbb{C}(\mathbb{R})/I \), \( \mathbf{M} \) any (separable) \( \mathbb{C} \)-manifold, \( \mathbf{I} \) any ideal. The reader is assumed to be familiar with the basic facts about \( \mathbb{C} \)-rings, which can e.g. be found in [B], [K], [MR1].

1.2. The smooth Zariski topos \( \mathbf{Z} \). Let \( \mathbb{C}(\mathbb{R}^n)/I \) be an object of \( \mathbf{L} \). A finite open cover of \( \mathbb{C}(\mathbb{R}^n)/I \) is a family of maps in \( \mathbf{L} \) of the form

\[
\{ \mathbb{C}(U_j)/I \}_{j=1}^k \to \mathbb{C}(\mathbb{R}^n)/I
\]

where \( U_j \subseteq \mathbb{R}^n \) are open subsets such that \( \mathbb{Z}(I_0) \subseteq U_1 \cup \cdots \cup U_k \) for some finitely generated ideal \( I_0 \subseteq I \), \( \mathbb{Z}(I_0) \) denotes the zero set of \( I_0 \), and the maps in [1] are induced by the inclusions \( U_j \subseteq \mathbb{R}^n \).

Recall that the category \( \mathbf{L} \) has finite products, given by

\[
\mathbb{C}(\mathbb{R}^n)/I \times \mathbb{C}(\mathbb{R}^m)/J \cong \mathbb{C}(\mathbb{R}^n \times \mathbb{R}^m)/(I,J).
\]

(1, J) being the ideal generated in the obvious way by \( I \) and \( J \).

2. We define a Grothendieck topology on \( \mathbf{L} \) by taking as covering families of an \( \overline{A} \in \mathbf{L} \) all families of the form

\[
\{ \mathbb{C}(\overline{A} \times B)/I \}_{i=1}^n \to \mathbb{C}(\overline{A})/I
\]

where \( \{ \mathbb{C}(\overline{A} \times B)/I \}_{i=1}^n \) is a finite open cover of \( \overline{A} \times \overline{B} \), and \( \overline{B} \neq 0 \) (i.e. \( B \) is not the trivial ring). In other words, the Grothendieck topology on \( \mathbf{L} \) is the one generated by the open covers as in [1], and by the singleton covers \( \{ \mathbb{R} \to 1 \} \), where \( 1 \) is the terminal object of \( \mathbf{L} \) and \( \overline{B} \neq 0 \). Note that the covers described by [1] are stable under pullback, and also under composition in the (weaker than usual) sense that if \( \{ \mathbb{C}(\overline{A} \times B)/I \}_{i=1}^n \to \mathbb{C}(\overline{A})/I \) and \( \{ \mathbb{C}(\overline{A})/I \}_{j=1}^m \) are covers of the type in [3] then there is a cover \( \{ \mathbb{C}(\overline{A} \times B)/I \}_{i=1}^n \) of the type described in [3] which refines the family composites \( \{ \mathbb{C}(\overline{A} \times B)/I \}_{i=1}^n \). It is not difficult to see that this Grothendieck topology is subcanonical (for the finite open covers, one uses a standard partition-of-unity argument).

The smooth Zariski topos \( \mathbf{Z} \) is the topos of sheaves on \( \mathbf{L} \) for the Grothendieck topology defined by the covers as in [3]. (Remark: in [HR3], we only considered finite open covers, thus defining a different topos. Here we also include projections \( \overline{A} \to \overline{A} \) as singleton covers (\( \overline{B} \neq 0 \), so as to force the existence of invertible infinitesimals, or more generally, an "internal Nullstellensatz", see 1.6 below.)

1.3. The embedding \( s: \mathbf{M} \to \mathbf{Z} \) of manifolds. The functor

\[
\mathbf{M} \to \mathbf{L}, \quad \mathbf{M} \to \mathbb{C}(\overline{M})
\]

defines a full embedding of the category \( \mathbf{M} \) of (smooth, separable) manifolds into the category \( \mathbf{Z} \), which preserves transversal pullbacks ([D]). We write \( s \) for the composite \( \mathbf{M} \to \mathbf{L} \to \mathbf{Z} \), where \( Y \) is the Yoneda embedding, which maps into \( \mathbf{Z} \) by subcanonicality of the Grothendieck topology. So

\[
s: \mathbf{M} \to s(\mathbf{M}) \to \mathbf{L} \to \mathbb{C}(\overline{M})
\]

also defines a full embedding, preserving transversal pullbacks. In fact, the objects \( s(\mathbf{M}) \in \mathbf{Z} \) in the image of \( s \) are "internal manifolds" in \( \mathbf{Z} \), and can be equipped with a canonical topology (not Grothendieck topology), see 2.1-2.5 below.

1.4. The line \( \mathbb{R} \). This is the object \( \mathbb{R} = s(\mathbb{R}) \) of \( \mathbf{Z} \). As a sheaf on \( \mathbf{L} \), \( \mathbb{R} \) can equivalently be described as

\[
\mathbb{R} = \text{the underlying set of } \mathbb{A} \quad (\mathbb{A} \in \mathbf{L}).
\]

\( \mathbb{R} \) is a commutative ring object in \( \mathbf{Z} \), with a canonical order \( < \) given by \( s(\mathbb{R}_0) = R_0 \subseteq \mathbb{R} \). In other words, for \( a, b \in s(\mathbb{R}) \), \( a = c(\mathbb{R})/I \)

\[
\mathbb{A} - a < b \quad \text{iff} \quad \exists x \in I(0) \ a(x) < b(x), \quad \text{for some finitely generated } I_0 \subseteq I \quad [1]
\]

\( \mathbb{R} \) is a local ring in \( \mathbf{Z} \), i.e. the statements \( 0 \neq 1 \) and \( \forall a, b, c \in s(\mathbb{R}) \ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \)

hold in \( \mathbf{Z} \), where \( s(\mathbb{R}) = \{ a \in \mathbb{R} \mid 3y \forall y x = 1 \} \).

1.5. The smooth natural numbers \( \mathbb{N} \). \( \mathbb{N} \) is the image \( s(\mathbb{N}) \) of the natural numbers, regarded as a discrete manifold. So for \( A = \mathbb{C}(\mathbb{R}^n)/I \), \( s(\mathbb{N}) \) is the set of equivalence classes of locally constant functions \( f: \mathbf{U} \to \mathbf{N} \), where \( \mathbb{Z}(I_0) \subseteq U \subseteq \mathbb{R}^n \) some finitely generated ideal \( I_0 \subseteq I \), and \( f: \mathbf{U} \to \mathbf{N} \) is equivalent to \( f': \mathbf{U}' \to \mathbf{N} \) if \( f' \) coincides with \( f \) on a neighbourhood of \( \mathbb{Z}(I_1) \) for a possibly bigger finitely generated ideal \( I_0 \cup I_1 \subseteq I \). It is \( \mathbb{N} \), rather than the natural number object of \( \mathbf{Z} \), that we will use to define notions in \( \mathbf{Z} \). For algebraic notions this was extensively discussed in [HR3], but it applies also to topology. For example, to define compactness we use finiteness relative to \( \mathbb{N} \); an object \( S \in \mathbf{Z} \) is \( s \)-finite if (it holds in \( \mathbf{Z} \) that) there
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But \( A \times N \ni a(U_j) \rightarrow P (v_j, p_j) \) by \([4]\), so by restricting along \( I \cap s(V_{i,j}) \xrightarrow{\pi_1} (A \times N) \cap s(U_j) \) we conclude that

\[
I \cap s(V_{i,j}) \rightarrow P (v_j, p_j) \cdot \pi_1.
\]

Moreover, if \((x, n, m) \in V_{i,j} \), then \( p_j(x, n) \leq \lambda (x, n) \cdot b(x, n) \), so \([8]\) follows from \([9]\). We conclude that

\[
A \cap V_0 \ni \exists \in \alpha \cdot v_0 \leq n_0 \in b \cdot p (n, m).
\]

Since \( A = c \times c^* \ni c \) is a cover, and \( R \) is Archimedean (cf. 1.7.(1) below), \((1)\) follows.

This proves the validity of \((1)\) in \( Z \).

1.6. Infinitesimal spaces in \( Z \). There are the usual infinitesimal spaces

\[
D_k(n) = C^n_c(R^n) / (\beta^n_0) = k \in k + 1, k \in k^n_0, k^n_0 = 0, \text{ all } n \text{ with } |n| = k + 1, \text{ used to formulate the Kock-Lawvere axiom and its generalizations} \text{ (see [K]). The space of infinitesimal elements of } R \text{ is}
\]

\[
\Delta = \bigwedge_{n \in \mathbb{N}} \left[ -\frac{1}{n+1}, \frac{1}{n+1} \right] = C_0^0(R),
\]

and the space of invertible infinitesimals is

\[
\Pi = \Delta \setminus \{0\} = C^\infty_c(R^n) / (n \in \mathbb{N}) \cdot R^n,
\]

where \( R^* = R \setminus \{0\} \), and \( n \in \mathbb{N} = \{f \mid |f| \geq 0, \text{ all } f \in k^n_0 \} \). Since the ring \( C^\infty_c(R^n) / (n \in \mathbb{N}) \cdot R^n \) is nontrivial, \( \Pi \rightarrow 1 \) is a cover, and hence invertible infinitesimals exist in \( Z \) in the sense that

\[
Z \ni 3a(x \in \mathbb{N}).
\]

(More generally, any non-trivial locus \( 0 \) has a point in \( Z \), in the sense that \( 1 \ni 3a(x \in \mathbb{N}) \).)

1.7. Some axioms of synthetic differential geometry. As said, \( R \) is a local ring object in \( Z \). \( R \) is Archimedean in the sense that

\[
\forall \alpha \in R \exists n \in \mathbb{N} \times a \leq n
\]

holds in \( Z \). Other axioms that hold in \( Z \) are for example the Kock-Lawvere axiom, and the integration axiom. Moreover, \( (-)^P, Z \rightarrow Z \) has a right-adjoint (in fact, \( (-)^P \) has a right adjoint for any dual \( W \) of a Weil algebra \( W \)). For more on \( Z \) as a model of synthetic differential geometry, see [HR3].
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2. Manifolds in the smooth Zariski topos: Applications of Ostrand’s theorem

In this section, we will show how topological properties of manifolds carry over to $\mathcal{Z}$ via the embedding $\mathcal{M} \rightarrow \mathcal{Z}$. An essential tool in transferring such properties to $\mathcal{Z}$ is the following result from dimension theory.

Ostrand's theorem. Let $M$ be a $d$-dimensional manifold, and let $U = \{U_n\}_{n \in \mathbb{N}}$ be a locally finite open cover of $M$. Then there exists an open refinement $V$ of $U$ such that $V = \{V_{n_1,m_1}, \ldots, V_{n_d,m_d}\}$ where each $V_{n,m}$ is a family $\{V_{i,m,n} : i \in \mathbb{N}\}$ of pairwise disjoint open sets such that $V_{i,m,n} \subset U_m$ (all $m \in \mathbb{N}$, all $i = \{1, \ldots, d\}$).

This formulation of Ostrand’s theorem is not the strongest one possible, but it suffices for our purposes. For a stronger version, and a proof, see [O] or [E].

Before anything else, we have to explain how for a manifold $M$, $s(M)$ is a topological space object of $\mathcal{Z}$. The definition given in 2.1 below may seem rather complicated, but it appears to be the most workable one. For some more justification, see also remark 2.5.

2.1. The internal topology of manifolds. We define a basis for the internal topology on $s(M)$, where $M$ is a manifold, by letting $s(M)$ be the smallest internal topology containing all the subobjects of stage $N \in \mathbb{E}$ given by $s(U \cup U \times \{n\}) \in s(M) \times N$, $n \in \mathbb{N}$, $\forall n \in \mathbb{N}$.

More explicitly, we can describe a basic open $U$ at stage $\bar{A}$ by a map $\bar{A} \rightarrow [0,1]$ and a sequence of open $\{U_n\}_{n \in \mathbb{N}}$ in $M$. This defines a subsheaf $U \in P(s(M))(\bar{A})$ as follows. For $\bar{B} \rightarrow \bar{A}$, $\bar{B} \rightarrow s(M)$, $\bar{B} \rightarrow U$ iff there is a finitely generated $J_0 \subset J$ (where $B = \mathbb{R}^n(J_0)/J$) and an open $0 \in \mathbb{Z}(J_0)$ such that $p \in f$ is represented by a map $0 \rightarrow \mathbb{R}$ with value $P_k$ on $0 \subset 0$ say, and $a$ is represented by $0 \rightarrow \mathbb{R}$, with $a(B) \subset U_k$. (Note that by the tubular neighbourhood theorem, we can always represent a map $\bar{B} \rightarrow s(M)$ by a smooth $a(x) : \bar{U} \rightarrow \bar{M}$, where $U \in \mathbb{Z}(J_0)$ for some finitely generated $J_0 \subset J$, and $0$ open.)

This definition seems perhaps a little bizarre. But let us remark that for the line $R = s(\mathbb{R})$, this gives the internal order topology having as a basis the set of intervals $\{(a,b) \mid a < b \in \mathbb{R} \}$. (Since $R$ is Archimedean, one can equivalently describe this by only taking $a$-rational open intervals $(p,q)$, $p < q$, $p, q \in \mathbb{Q}$, where $q = s(q)$ is the sheaf of $a$-rational numbers, defined just like $\mathbb{R}$ but with the discrete rationals instead of the natural numbers). Similarly, for the case $\mathbb{R}^n = s(\mathbb{R}^n)$, the topology defined above yields the product topology obtained from the order topology on $R$. So for the spaces $\mathbb{R}^n$, we have defined the correct internal topology. For general manifolds, see 2.5 below, where it will also be proved that we have indeed defined a basis for an internal topology!

2.2. Terminology. Let $\bar{A} \in \mathbb{E}$, $A = C^\infty(\mathbb{R}^n)/I$, and let $M$ be a manifold. Suppose

$\langle 0 \rangle \subset M$ is a cover of $\mathbb{Z}(I_0)$ for some finitely generated $I_0 \subset I$ by disjoint open subsets of $\mathbb{R}^n$, and let $\{U_m\}_{m \in \mathbb{N}}$ be a sequence of open subsets of $M$. Then these two sequences represent a unique internal basic open $U$ of $s(M)$ at stage $\bar{A}$ (namely defined in 2.1 by the sequence $\{U_m\}_{m \in \mathbb{N}}$ and the map $\mathbb{R} \rightarrow s(M)$ represented by $p(x) : U_m \rightarrow \mathbb{N}$, $p(x) = m \iff x \in U_m$), and every internal basic open is of this form. We will say that $U$ is the open given by $U_m$ over $U_m$, or by $U_m$ at $U_m$, or with value $U_m$ over $U_m$.

2.3. Generic elements of basic opens. Let $\bar{A} \in \mathbb{E}$ and let $U$ be a basic open of $s(M)$ at stage $\bar{A}$, given by $U_m$ over $U_m$, just as in 2.2. Let $V = U \cap (U_m \times N)$, $m \in \mathbb{N}$, and let $B = \mathbb{R}^n(V)/(I)$, where $(1)$ is the ideal generated by the functions $\{1\} : U \rightarrow \mathbb{R}$, for $i \in I$. Then $\pi_1 : V \rightarrow \mathbb{R}^n$ induces a map $\bar{B} \rightarrow \bar{A}$, and $\pi_2 : V \rightarrow M$ gives an element $\bar{B} \rightarrow s(M)$ of $s(M)$ at stage $\bar{A}$. Clearly $\bar{B} \parallel \pi_2 \in U$. Moreover, in the generic element of $U$, in the usual sense: if $\bar{C} \rightarrow \bar{A}$ is any map in $\mathbb{E}$, and $\bar{C} \rightarrow s(M)$ is an element of $s(M)$ with $\bar{C} \parallel \pi_2 \in U$, then $x$ is a restriction of $\pi_2$, i.e. there is a $g$ making the following diagram commute.

Next we wish to say that $s : \mathbb{M} \rightarrow \mathcal{Z}$ preserves open covers. However, since the site $\mathcal{Z}$ for $\mathcal{Z}$ is defined using finite covering families only, it is clear that we cannot take this to mean: "if $\{U_i\}_{i \in I}$ is an open cover of a manifold $M$, then $\{s(U_i)\}_{i \in I}$ is an epimorphic family in $\mathcal{Z}". This simplicity interpretation (sometimes used as a requirement on models for synthetic differential geometry) is unnecessarily strong. Let us consider countable covers only, which is not really a restriction since we assumed manifolds to be separable. (In fact, everything we will say extends to arbitrarily indexed covers, by defining the embedding $s : \mathbb{M} \rightarrow \mathcal{Z}$ also for (at least discrete) manifolds of arbitrary cardinality.) For an open cover $\{U_m\}_{m \in \mathbb{N}}$ of $M \in \mathcal{M}$, we obtain a corresponding open $U$ of $s(M)$ at stage $\bar{A}$ (cf. 2.1), and hence a map $\bar{N} \rightarrow s(M)$, i.e., an internal s-countable (indexed by the smooth natural numbers) family of open sets. We claim that this is an internal cover.

2.4. Proposition. The embedding $s : \mathbb{M} \rightarrow \mathcal{Z}$ preserves countable covers. More precisely, if $\{U_m\}_{m \in \mathbb{N}}$ is an open cover of $M$, then for the corresponding s-countable
family of opens \( N \overset{U}{\rightarrow} O(s(M)) \) in \( Z \) we have
\[ Z \models \forall x \in s(M) \exists n \in N \times \forall n_k \in \mathbb{N}. \]

Proof. To show that \( Z \models \forall x \in s(M) \exists n \in N \times \forall n_k \in \mathbb{N} \), it suffices to consider the generic \( \gamma \in s(M) \) at stage \( s(M) \) given by the identity, and show that \( s(M) \models \exists n \in N \times \gamma \in U_n \). To this end, let \( \{ V_n \} \) be a neighbourhood finite refinement of the given cover \( \{ U_n \} \) with \( V_k \subseteq U_k \). Say by Ostrand's theorem, there is a refinement of the form \( V \models \forall \gamma \in \gamma^d \).

\[ \gamma = \{ \gamma^d \} \]

(\( d = \text{dim}(M) + 1 \)), where \( U^d \overset{\gamma^d}{\rightarrow} V^d \) is a family of pairwise disjoint opens, and \( V^d_k \subseteq V_n \). Let \( \gamma^d \rightarrow V^d \). At stage \( s(M) \), we define a smooth natural number \( n_k \) with value \( n_k \) over \( \gamma^d_k \). Then it is clear from \( \gamma^d \) that \( s(M) \models \gamma \in U_n \), since \( U_n \) is the internal (basic) open given by \( U_n \) over \( V^d_k \). Since the \( V^d_k \) form an open cover of \( M \), we conclude \( s(M) \models \exists n \in N \times \gamma \in U_n \).

2.5. Remark. We are now in a position to give some more justifications for the definition 2.1 of the topology on \( s(M) \). Synthetically, one would construct a "manifold" inside \( Z \) by patching together copies of \( R^d \). If \( M \) is a manifold in \( Z \), and \( U_n \) is a countable atlas for \( M \) with diffeomorphisms \( q_n : R^d_n \rightarrow R^d \), then as in 2.4, \( Z \models s(M) = U V \) (a union over smooth natural numbers), and working inside \( Z \), we can give \( s(M) \) the weak topology with respect to this \( s \)-countable cover \( U_n \in N \), where each \( U_n \) is given the topology obtained from the internal topology on \( R^d \) induced by the order topology on \( R \) and the (external) diffeomorphisms \( q_n : R^d_n \rightarrow R^d \in N(\gamma) \). The resulting topology is the same:

Proposition. The internal topology on \( s(M) \) as defined in 2.1 coincides with the weak topology given by an \( s \)-countable atlas \( U_n \in N \) of copies of \( R^d \) as just described.

We omit the proof, which is a little tedious, but not really difficult.

Corollary. For every \( M \in N \), the basic opens defined in 2.1 form indeed a basis for a topology, making \( s(M) \) into an \( s \)-topological space, i.e., the intersection of \( s \)-finitely many opens is again open.

Proof. We only need to consider the case of the order topology on \( R = s(M) \), since \( R^d \) has the product topology induced by this topology on \( R \), and the general case then follows from the preceding proposition. For \( R \), reason in \( Z \), and suppose

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for example that we are given basic open neighbourhoods \( (U, i = n) \) of \( U \in N \). By Archmedeaness, \( \forall i \in N \exists k \in N \left( \frac{1}{k + 1}, \frac{1}{k + 1} \right) \subseteq U \). By bounded search (see 1.5) we find an upper bound \( k_0 \) for the \( k \)'s needed, i.e. \( \forall i \in N \left( \frac{1}{k + 1}, \frac{1}{k + 1} \right) \subseteq U \).

So \( \bigwedge_i U_i \) is also a neighbourhood of \( 0 \) in \( N \).

A topological space \( X \) in \( Z \) is \( s \)-compact if every open cover of \( X \) has an \( s \)-finite refinement. More precisely, for every open cover \( U \) of \( X \) there exists an \( n \in N \) and a map \( \{ n(n) \} \rightarrow \gamma \) such that \( X = U \overset{V \rightarrow N}{} \) and \( \forall n \in N, \exists n \in N \).

2.6. Theorem. The embedding \( s(M) \rightarrow Z \) preserves compactness. In other words, if \( M \) is compact then \( Z \models s(M) \) is \( s \)-compact, for every manifold \( M \).

Proof. Let \( M \) be a compact manifold, let \( U \) be an open cover at stage \( \gamma \), and assume \( M \) is of basic open. Since \( \gamma \rightarrow \gamma \), \( \gamma \models \gamma \)

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for a basic open \( U \) at stage \( \gamma \), \( i = 1, \ldots, 0 \). By choosing \( i = 0 \) and \( i = 0 \) adequately, we may assume that \( U \) is given by an open \( U \in M \) over \( 0 \), where \( (0) \)

is a disjoint cover of \( 0 \) (cf. 2.2), and since \( \gamma \rightarrow \gamma \), \( \gamma \)

the map \( 0 \) maps \( 0 \) into \( U \).

By compactness of \( M \), we can find an open cover \( \{ U_n \} \) of \( Z(\gamma) \) and for each \( N \) open subsets \( E_n \) of \( M \) such that \( E_n \in U \in M \), and each \( E_n \in E \in M \) is contained in some \( U_n \), say \( N \in U \) (cf. 2.2), and since \( \gamma \rightarrow \gamma \), the map \( 0 \) maps \( 0 \) into \( U \).

Moreover, by passing to a suitable refinement, we may without loss assume that the cover \( \{ U \} \) is neighbourhood finite.

An application of Ostrand's theorem now yields a refinement of \( \{ U \} \) of the form \( \gamma \overset{\gamma}{\rightarrow} U \), where \( \gamma \) is a family of pairwise disjoint open sets, \( \gamma \rightarrow \gamma \), with \( \gamma \in \gamma(n, \ldots, n) \).

Let \( \gamma \rightarrow \gamma \), define a smooth
natural number \( q \), at \( s(W^r) \), i.e. a locally constant map \( \bar{s}^r \to \mathbb{N} \), by

\[ q \bar{s}^r_n = k_n. \]

Also, we can define a family of basic opens \( (V_q \mid q \leq q^r) \) of \( s(M) \) at stage \( s(W^r) \), as follows. The generic \( q \leq q^r \) at stage \( W^r \) is the projection \( s(U_W \times (0, \ldots, k_n)) \to \mathbb{N} \) (see 2.3), and a family \( (V_q \mid q \leq q^r) \) of basic opens of \( s(M) \) at stage \( s(W^r) \) is the same as a single open \( V \) at stage \( s(W^r) \times (0, \ldots, k_n) \). \( V \) is defined as

\[ V = s^r_n \text{ over } \bar{s}^r \times (j). \]

(c.f. 2.2).

Let \( A^r = s(W^r) \cap \bar{A} = C(W^r) / (I \bar{W}^r) \). Then \( q^r \) restricts to an element of \( N \) at stage \( A^r \), and \( V = (V_q \mid q \leq q^r) \) restricts to a family of opens of \( s(M) \) at stage \( A^r \). We claim that

\[ A^r \models \forall \theta \in \mathfrak{s}(M) \exists q \in N(1 \leq q \leq \bar{W}^r \text{ with } V \). \]

(1)

\[ A^r \models \forall \theta \in \mathfrak{s}(M) \exists q \in N(1 \leq q \leq \bar{W}^r \text{ with } V \). \]

(2)

Since \( A^r = \bar{A} = \bar{A}_1 \times \bar{A}_2 \overset{\pi_1}{\to} \bar{A}_1 \overset{\pi_1}{\to} A^r \) is a cover of \( \bar{A}_1 \) in \( L \), [1] and [2] imply that

\[ \bar{A}_1 \models " \text{U has an } \sigma \text{-finite refinement}" \]

which would complete the proof. It thus remains to show [1] and [2].

To prove [1], we only need to consider the generic \( x \in \mathfrak{s}(M) \) and show that

\[ A^r \times \mathfrak{s}(M) \models \exists q \leq q^r \exists \theta \in \mathfrak{v} \).

(3)

Fix \( \theta \), and let \( \bar{A}^r = U \bar{A}^r \times \bar{W}^r \times (n, 1), 1(n, j) = 1 \). Then \( A^r \times \mathfrak{s}(M) = \bar{A}^r \cup \bar{A}^r \).

Moreover, \( \bar{A}^r \) is a union of disjoint sets, so we may define a smooth natural number \( q^r \) at \( s(W^r) \) by

\[ q^r = \text{on } \bar{s}^r \times E_n^r \]

Then the basic open \( V^r \) of \( s(M) \) at \( s(W^r) \) is the open given by \( E^r \) over \( \bar{s}^r \times E_n^r \), so it is clear that \( s(W^r) \cap \bar{A}^r \times \mathfrak{s}(M) \models \exists \theta \in \mathfrak{v} \).

Since \( \bar{A}^r \times \mathfrak{s}(M) = \bar{A}^r \cup \bar{A}^r \), we conclude that [3] holds.

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Finally, we prove [2]. It suffices to take the generic \( q \leq q^r \), i.e. to show

\[ A^r \models \exists \theta \in \mathfrak{v} \]

(4)

where \( \bar{A} = (\bar{A} \cap \mathfrak{v}) \cap (\bar{A} \cap E^r) \). Similar to the \( \bar{A}^r \), define open sets \( \bar{A}^r = \bar{A} \cap \mathfrak{v} \times (1, \ldots, k_n) \). Since \( \bar{A}^r = \bar{A} / (\bar{W}^r \times (1, \ldots, k_n)) \), [4] follows if we can show that for each \( 1, 1(n, j) \),

\[ A^r \models \exists \theta \in \mathfrak{v} \]

(5)

To see that [5] holds, let \( \bar{A}^r = U \bar{A}^r \times (j) \times E_n^r \), and let \( \bar{A}^r \cap (\bar{A}^r \cap \mathfrak{v}) \). Then there are projections \( \bar{A}^r \xrightarrow{\pi_1} \bar{A}^r \) and \( \bar{A}^r \xrightarrow{\pi_2} \bar{A}^r \cap \mathfrak{v} \).

Since \( \pi_2 \circ \pi_1 \circ \pi_1 \) obviously has a right-inverse, so does \( \bar{A}^r \xrightarrow{\pi_1} \bar{A}^r \), and therefore [5] would follow if we show

\[ \bar{A}^r \models \exists \theta \in \mathfrak{v} \]

(6)

But \( \bar{A}^r \xrightarrow{\pi_1} \bar{A} \cap \mathfrak{v} \) factors through \( \bar{A} \) and \( \bar{A} \models \exists \theta \in \mathfrak{v} \), where \( \bar{A} \), is given by \( \bar{A} \in \mathfrak{v} \) over \( \bar{A} \cap \mathfrak{v} \), and moreover \( \bar{A} \models \exists \theta \in \mathfrak{v} \), as we saw in the beginning of the proof. Restricting this along \( \bar{A}^r \xrightarrow{\pi_1} \bar{A}^r \), we find that \( \bar{A} \models \exists \theta \in \mathfrak{v} \), where \( \bar{A} \) is given at \( \bar{A}^r \) by \( \bar{A} \cap \mathfrak{v} \) over \( \bar{A} \times (j) \times E_n^r \) and \( \bar{A}^r \) is given at \( \bar{A}^r \) by \( \bar{A} \cap \mathfrak{v} \) over \( \bar{A}^r \times (j) \times E_n^r \). Since \( \bar{A} \cap \mathfrak{v} \cap E_n^r \), and \( \bar{A} \cap \mathfrak{v} \cap E_n^r \), we conclude that \( \bar{A}^r \models \exists \theta \in \mathfrak{v} \), which proves [6], and completes the proof of the theorem.

Remark. In [33], the special case of 2.6 was proved for \( M = [0, 1] \subseteq \mathbb{R} \), by a different argument not involving Ostrand's theorem, and for another topos, realy (cf. the remark at the end of 1.2 above). As we pointed out there, \( s \) does not preserve compactness if we interpret compactness in the standard way and require every open cover to have a (Kuratowski) finite subcover.

2.7. Theorem. Let \( M \) and \( N \) be manifolds. In \( Z \) it holds that every function \( s(M) \to s(N) \) is continuous.
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The proof of 2.8 is easy application of Ostrander's theorem, which we omit. More difficult and more interesting is the following result.

2.9. Theorem. Let \( M \) be a manifold. Then it is valid in \( Z \) that for every open cover \( U \) of \( M \) there exists an \((s\) - countable\) \(- s\)-partition of unity \( \{p_n\}_{n=1}^\infty \) subordinate to \( U \).

Proof. Let \( U \) be a cover of \( \sigma(M) \) at a given stage \( \bar{A}_1 \), and assume that \( U \) consists of basic open sets. Then \( \bar{A}_1 \times \sigma(M) \models U \models (\forall \alpha \in \mathcal{A}_2 \forall \beta \in \mathcal{B}_2 \models U) \), and this implies (as in the proof of 2.6) that there is an \( \bar{A}_2 \neq \emptyset \) such that, if we write \( \bar{A} = \bar{A}_1 \times \bar{A}_2 \subset \sigma(M)/I \), there is a finitely generated \( I \subset I \) and a cover \( \{U_{01}, \ldots, U_{0T}\} \subset Z(\bar{I}) \times M \) by open subsets of \( \mathbb{R}^n \) \times \mathcal{M} \) with the property that there is a basic open \( U_1 \) of \( \sigma(M) \) at stage \( s(01) \) given by \( U_{0m} \subseteq \sigma(M) \) over \( 0_{0m} \) say, where \( \{0_{01}, \ldots, 0_{0T}\} \) is a disjoint cover of \( 0_{0} \) with \( \pi_2(0_{01}) \subset U_{01} \); moreover, \( \sigma(0_{01}) \cap A \models \varphi \). Let \( \{p_n\}_{n=1}^\infty \) be a partition of unity subordinate to \( \{U_n\}_{n=1}^\infty \) on \( \mathbb{R}^n \times \mathcal{M} \), such that for each \( n \in \mathbb{N} \), \( \sum_{n=1}^\infty p_n(x) = 1 \) for all \( x \in \mathbb{R}^n \times \mathcal{M} \), and \( \text{supp}(p_n) \subset U_n \). Then \( \{p_n\}_{n=1}^\infty \) defines a map \( N \times \bar{A} \rightarrow \sigma(M) \models \{0_{01}\} \) in \( Z \), and we claim that

\[ \bar{A} \models \varphi \text{ is an } s\text{-partition of unity subordinate to } U_n \text{, i.e.,} \]

\[ \bar{A} \models \forall x \in \sigma(M) \forall y \in U \models (x \in U \forall p_n(x) = 0) \quad (1) \]

\[ \bar{A} \models \forall x \in \sigma(M) \exists \forall n \in \mathbb{N} \exists n \in \mathbb{N} \models (x \in U \forall p_n(x) = 0) \quad (2) \]

\[ \bar{A} \models \forall x \in \sigma(M) \forall p_n(x) = 1 \quad (3) \]

To prove (1), we consider the generic \( n = \pi_2 \), and show

\[ \bar{A} \times N \models \forall U \forall x \in \sigma(M) \forall y \in U \exists p_n(x) = 0 \quad (4) \]

Cover \( \bar{A} \) by \( \bar{A}_1 \times \{1, \ldots, 0\} \), where \( \bar{A}_1 \ni n \sigma(U_{01}) | n = 1 \), and define basic opens \( E_n \) of \( \sigma(M) \) at \( \bar{A}_1 \times N \) by \( E_n = U_{0m} \) over \( \bar{A}_1 \times \{n\} \). Then

\[ \bar{A}_1 \times N \times \sigma(M) \models (n \models \exists E_n \forall p_n(x) = 0) \quad (5) \]

for \( \pi_2 \in E_n \) holds on \( U_{\mathcal{D}_n \times \{n\} \times E_n} | n = 1 \), and \( \text{supp}(p_n) \subset D \times E_n \). By genericity of \( \pi_2 \), [5] gives that

\[ \bar{A}_1 \times N \models \forall x \in \sigma(M) \forall y \in U \forall p_n(x) = 0 \quad (6) \]
so [4] follows, provided we can show that
\[ \bar{A}_1 x N \models E_1 \in U. \]  
To this end, consider the projection
\[ U (D \times \mathbf{R})_n \rightarrow U D_n. \]  
Obviously, this map locally has a right-inverse, defined over each open set \( D_n \). By Ostrand’s theorem applied to a locally finite refinement of the cover \( \{ D_n : n \in \mathbb{N} \} \) of \( U D_n \), we find

\[ \text{finitely many opens } S_1^{i_1} \ldots S_k^{i_k} \text{ such that } \bigcup_{i_1 \leq i \leq j} S_j \rightarrow U D_n, \]  
and sections \( t_j : S_j^{i_1} \rightarrow S_{j+1}^{i_1} \) of \( S_k \). Since the \( S_j^{i_1} (1 \leq j \leq r) \) induce a cover of \( \bar{A}_1 \), [7] would now follow if we show
\[ \bar{B}_1 x N \models E_1 \in U \]  
where \( \bar{B}_1 = (\bar{A}_1 \times s(M)) \cap \bigcup (D x \mathbf{R})_n \), considered as an object over \( \bar{A}_1 \) via the projection
\[ \bar{B}_1 \rightarrow \bar{A}_1. \]  
But \( \bar{B}_1 \models E_1 \in U \), since \( \bar{A}_1 \models U_1 \in U \), and at \( \bar{B}_1 \), \( E_1 \) and \( U_1 \) are both given by the open \( U_{1n} \) over \( D_n \) as \( E_n \times \{ n \} \). Thus \[ 8 \] holds, which completes the proof of [1].

To show [2], we can take the generic \( x \in s(M) \), i.e., show that
\[ \bar{A} x \times s(M) \models (\exists y \in U \forall z_1 \ldots z_n \in U \forall \eta \in s(M)) \]  
where \( \bar{A} = \bar{A}_1 \times s(M) \cap \bigcup (D x \mathbf{R})_n \), considered as an object over \( \bar{A}_1 \) via the projection
\[ \bar{A} \rightarrow \bar{A}_1. \]  
But \( \bar{A} \models U_1 \in U \), since \( \bar{A}_1 \models U_1 \in U \), and at \( \bar{A} \), \( U_1 \) and \( U_1 \) are both given by the open \( U_{1n} \) over \( D_n \) as \( E_n \times \{ n \} \). Thus \[ 9 \] holds, which completes the proof of [1].

2.10. Corollary. Every manifold is \( s \)-Lindelöf in \( Z \), i.e., for any \( M \in \#M \) it holds in \( Z \) that every open cover \( U \) of \( M \) has an open refinement indexed by the smooth natural numbers \( \mathbb{N} \).

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Proof. Clear from 2.9 and 2.7.

We now turn to connectedness. A topological space \( X \) in \( Z \) is called connected if it holds in \( Z \) that \( \forall U, V \in \mathcal{O}(X) (\exists x \in U \forall y \in V \models x \neq y) \rightarrow U \cap V = \emptyset \). Then \( X \models \text{connected} \) if \( X \models \text{connected} \) in \( Z \).

2.11. Theorem. The embedding \( s : M \rightarrow Z \) preserves connectedness; i.e., \( Z \models \text{connected} \) if \( \forall M \in \mathcal{M} \) is connected.

Proof. We just give a sketch: suppose \( U \) and \( V \) are internal open at \( \bar{A} \) (not necessarily basic opens!) such that \( \bar{A} \models (3x \in U(x) \land 3x \in V(x) \land x \in \mathbb{R}) \). Then \( \bar{A} \models (\exists x \in U \land \exists x \in V \land x \in \mathbb{R}) \), so (replacing \( \bar{A} \) by \( \mathbb{R} \times \mathbb{R} \) if necessary) if we write \( A = C^n (\mathbb{R} \times \mathbb{R}) / \sim \), there is a finitely generated \( \mathbb{Z}_0 \subset \mathbb{R} \) and \( \forall A_0 \in \mathbb{R} \models \mathbb{R} \cap \mathbb{R} = \mathbb{R} \), and \( s(A_0) \cap A \times s(M) \models \rho_2 \in U \), resp. \( s(A_0) \cap A \times s(M) \models \rho_2 \in V \). Since \( \bar{A} \models 3x \in U \lor 3x \in V \), we may assume (choosing \( \rho_1 \) bigger and adding another non-zero factor \( \rho_1 \) if necessary) that for each \( x \in \mathbb{R}_0 \), \( U \cap (\{ x \} \times M) \neq \emptyset \). So by connectedness of \( N \), the projection \( \mathbb{R}_0 \rightarrow \mathbb{R}_0 \), maps onto a neighborhood of \( U_0 \). Using Ostrand’s theorem, we find a finite cover \( U_1 \ldots U_k \) of \( U_0 \) on which \( \rho_2 \) splits, i.e., there are \( U_1 \models \rho_1 \cap \rho_2 = \rho_2 \models (x) = x \). This will give \( \mathbb{R} \models (3x \times \mathbb{R}) \).

2.12. Remark. A topological space \( X \) is called chain-connected if for every open cover \( U \) of \( X \) and any two points \( x, y \in U \) there is a sequence \( U_0 \ldots U_n \) of elements of \( U \) such that \( x \in U_0 \), \( y \in U_n \), and \( \forall U \models \exists x \in U \models (3x \in U \land \mathbb{R} \cap \mathbb{R} = \mathbb{R}) \). Intuitionally, chain-connectedness is stronger than connectedness, and seems more appropriate for doing intuitionistic topology. (This was pointed out in the more general context of locales in the appendix of [M].)

However, this notion of chain-connectedness is not suitable for working in \( Z \). (For example, the unit-interval \([0,1] = [0,1] \subset \mathbb{R} \) cannot be chain-connected, since it is not compact.) Just as we replaced compactness by \( s \)-compactness, we may define a space \( X \) to be \( s \)-chain-connected if for every open cover \( U \) of \( X \) and any two points \( x, y \in X \) there is an \( s \)-finite sequence \( \{ U_n \} \in \mathbb{N} \), \( n \in \mathbb{N} \), of opens of \( X \) with \( x \in U_0 \), \( y \in U_n \), and \( \forall U \models (3x \in U \land \mathbb{R} \cap \mathbb{R} = \mathbb{R}) \). So, as with compactness, \( (\text{Kuratowski -}) \) finite is replaced by \( s \)-finite, and to avoid \( s \)-finite choice we require a sequence refining \( U \), rather than a sequence of elements of \( U \). By a relatively complicated and lengthy application of Ostrand’s theorem one can now show

2.13. Theorem. The embedding \( s : M \rightarrow Z \) maps connected manifolds to \( s \)-chain-connected manifolds.
We omit the proof.

As a final remark, we state a result of a slightly different nature, which can again be proved by using Ostrand's theorem.

2.14. Theorem. (Open refinement theorem) Let $M$ be a manifold. Then in $Z$ it holds that every $\omega$-countable cover of $M$ has an open refinement, i.e.,

$$Z \models \forall \phi \exists F(n)(\exists x \in \mathbb{N} \land \forall n \in F(n) \to \exists x \in \mathbb{N} \exists u \in \mathbb{N} \land x \in u \subseteq F(n)).$$

Acknowledgements. I would like to thank G. Reyes for many inspiring discussions.

During the preparation of this paper, I was supported by the Ministère de l'Éducation du Gouvernement du Québec, through its sponsoring of the Centre Interuniversitaire en Études Catégoriques.

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