Rings of Smooth Functions and Their Localizations, II

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Introduction

\( C^\infty \)-rings are rings of smooth functions on a manifold, and rings obtained from these by taking directed colimits and by dividing out by ideals. In Moerdijk & Reyes (1994), henceforth referred to as "part I", a precise definition of this category was given, and some of the basic properties of \( C^\infty \)-rings were discussed; in particular, local \( C^\infty \)-rings were studied. We will assume that the reader is familiar with the results of part I, and with the notational conventions from that paper. We hasten to point out, however, that most of what we need from part I is really some basic classical analysis, rephrased in the context of \( C^\infty \)-rings. (For example, if \( f : R^m \rightarrow R \) represents an element \( a \) of the \( C^\infty \)-ring \( A = C^\infty (M)/I \), the universal solution \( A[\{a\}^{-1}] \) of inverting \( a \) in the category of \( C^\infty \)-rings is \( C^\infty (U_p)/I|U_p \), where \( U_p = \{ x \in R^m | f(x) < 0 \} \). This is not a new result, but essentially just the implicit function theorem.)

In this second paper, we will define and study the spectrum of a \( C^\infty \)-ring \( A \). If \( P \) is a prime ideal in a \( C^\infty \)-ring \( A \), it need not be true that the localization \( A_p \) (in the category of \( C^\infty \)-rings) is a local \( C^\infty \)-ring. This is true, however, for \( C^\infty \)-radical prime ideals, which were defined in part I. In section 1 of this paper we will construct the spectrum of a \( C^\infty \)-ring as the space of \( C^\infty \)-radical prime ideals, and show that it has universal properties similar to the usual spectrum of commutative algebra. In particular, the spectrum classifies the localizations of a \( C^\infty \)-ring \( A \) in the sense that for every localization of \( A \) there is a unique \( C^\infty \)-radical prime ideal \( P \) such that the given localization is isomorphic to \( A_p \). Still, \( A_p \) can be local, without \( P \) being \( C^\infty \)-radical. However, and we believe that this is the most interesting result of section 1, the equivalence does hold for rings of smooth functions on a manifold \( M : C^\infty (M)_P \) is local if and only if \( P \) is \( C^\infty \)-radical (theorem 1.14).

In section 2 we approach the construction of the spectrum of a \( C^\infty \)-ring from a topos theoretic viewpoint. We construct the spectrum of a \( C^\infty \)-ring in an arbitrary Grothendieck topos, and show that it has the right universal properties. In the case where the topos is the category of Sets, this construction coincides with

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one presented in section 1. As we remarked already in part 1, this generalization
of the construction of the spectrum to an arbitrary topos is necessary to show that
the spectrum as defined in section 1 is the correct one, since even in the case of
sets it can only be seen as constructing a universal localization in the category of
\( C \)-ringed toposes.

The motivation for investigating localizations of \( C \)-rings comes from the study of
models of Synthetic Differential Geometry, in particular, of the smooth version of
the Zariski topos (see Moerdijk & Reyes (1983)). This topos contains analogues of
schemes - "smooth schemes" - just as the usual Zariski topos contains counterparts
of the schemes of algebraic geometry (see Denzler & Gabriel (1970)). Furthermore,
this smooth Zariski topos allows the introduction of not only the usual (nilpotent)
infinities as in Synthetic Differential Geometry, but also of invertible infinitesimals
and infinitely large integers, as in Non Standard Analysis. This feature
forces us to consider arbitrary \( C \)-rings and their spectra, as opposed to the
approach taken in Dubuc (1981), who considers a more restricted notion of spectrum
corresponding to a model for Synthetic Differential Geometry where such features of
Non Standard Analysis are not present. In section 3 we will explain how his results
are obtained from ours, by restricting our attention to Archimedian localizations.
\( C \)-rings and their spectra, however, may be studied in their own right,
independently of Synthetic Differential Geometry. And our paper could be viewed in
this light.

Our spectrum has properties quite similar to the spectrum occurring in real algebraic
geometry (see Coste & Coste-Roy (1979), (1982)), but the presence of arbitrary
smooth maps, rather than just polynomial ones, allows us to bypass the problems
arising in the real algebraic case from the undefinability of order.

We point out that section 1 and the first half of section 3 can be read without any
knowledge of topos theory.

1. The spectrum of a \( C \)-ring

In this section we shall work exclusively in the classical context of the category
of sets.

1.1. Lemma. Let \( A \) be a \( C \)-ring, and \( P \) a prime ideal in \( A \). If \( P \) is
\( C \)-radical, then the \( C \)-ring \( A_P \) is a local ring. Moreover, if \( \pi : A \rightarrow A_P \)
denotes the universal \( C \)-homomorphism, then for each \( a \in A \), \( \pi(a) \) is invertible
iff \( a \notin P \). (Recall that \( A_P = \lim_{\rightarrow P} A \langle a^{-1} \rangle \) denotes the universal solution of
inverting all elements in \( A \setminus P \) in the category of \( C \)-rings, not in the category
of commutative rings.)

Rings of Smooth Functions

Proof. Writing \( A \) as the colimit of its finitely generated sub- \( C \)-rings, we
may assume \( A \) is itself finitely generated, say \( A = C^n(R) / I \), and since inverting
elements and quotienting by \( I \) commute, it suffices to consider the case
\( I = (0) \). So \( P \) is a \( C \)-radical prime ideal in \( C^n(R) \) and

\[
C^n(R)_P = \lim_{\rightarrow P} C^n(U_{\gamma})
\]

We claim that

\( \mathfrak{p} = \{ U_{\gamma} \mid F \notin U_{\gamma} \} \)

is a prime filter. Indeed, \( R = U_{\gamma} \), \( U_{\gamma} \cap U_{\gamma} = U_{\gamma} \), and if \( F \notin U_{\gamma} \) and \( g \) is
a characteristic function for \( V \), then \( Z(g) = Z(F) \), so \( f \notin P \) only if \( f \notin P \) since \( P \) is \( C \)-radical. So \( \mathfrak{p} \) is a filter. If \( U \cup V \in \mathfrak{p} \), say \( U \cup V = U_{\gamma} \) with \( f \notin P \), write \( V = U_{\gamma} \), \( W = U_{\gamma} \), then \( Z(f) = Z(g) \cap Z(h) = Z(g+h) \), so \( g^2 + h^2 \notin P \) since \( P \) is \( C \)-radical, hence \( g^2 \notin P \) or \( h^2 \notin P \), i.e., \( V \in \mathfrak{p} \)
or \( W \in \mathfrak{p} \), so \( \mathfrak{p} \) is prime.

But if \( \mathfrak{p} \) is any prime filter of open subsets of \( R \), \( \lim_{\mathfrak{p}} C^n(0) \) is obviously a
local ring.

Finally, we note that if \( a \in A \) and \( \eta(a) \) in \( A_{\mathfrak{p}} \) is invertible, then \( U_{\gamma} \supset U_{\gamma} \) for
some \( f \notin P \). So \( a \notin P \) since \( P \) is \( C \)-radical.

Contrary to the case of commutative rings, lemma 1.1 need not be true for arbitrary
prime ideals, as the following example shows.

1.2. Example. Let \( \mu = \mu_{\mathfrak{p}} \in (R) \) be the ideal of functions which are flat at \( 0 \),
i.e., \( \mu = \ker(T_0 : R \rightarrow \mathcal{D}([\mathbb{R}])) \). This is a non- \( C \)-radical prime ideal. Let
\( x : C(R) \) be the identity function. Then in \( C(R)^{\mathfrak{p}} / C(R^{\mathfrak{p}}(0)) \), consider
\( f, g \)

\[
f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}
\]

\( f + g = 1 \) in \( C(R^{\mathfrak{p}}(0)) \), hence in \( C(R^{\mathfrak{p}}) \). But neither \( f \) nor \( g \) can be inverting
in \( C(R^{\mathfrak{p}}) \). For \( f \), \( g \) were invertible, say, then there would be an
\( h \in \mu \) with \( f \supset U_0 \) \( \mu \), i.e., \( h \in \mu \), contradicting \( h \notin \mu \). So \( C(R^{\mathfrak{p}}) \)
is not a local \( C \)-ring.

If \( P \) is a countably generated prime ideal in \( C^{\mathfrak{p}}(R) \), then \( C^{\mathfrak{p}}(R)_P \) is always
a local ring. This is because such an ideal must be maximal (hence \( C \)-radical).

1.3. Theorem. Let \( M \) be a manifold, and let \( C(M) \) be the ring of smooth real-
valued functions on \( M \). Then every countably generated prime ideal in \( C(M) \) is
maximal.
For notational convenience, we only prove the case $M = \mathbb{R}$. The proof easily generalizes to arbitrary manifolds. We use the following lemmas.

Lemma 1: If $P \in C^{0}(\mathbb{R}^{n})$ is a prime ideal, then the filter 
\[ \mathcal{Z}(f) \setminus \{f\} \]
of zerosets is a prime filter.

Lemma 2: Let $F$ be a countably generated prime filter of closed sets in $\mathbb{R}^{n}$, then 
\[ \mathcal{N} \] consists of a single point.

Proof. Clearly $|\mathcal{N}| \leq 1$. The problem is to show $\mathcal{N} = \emptyset$. Let $(P_{n})_{n}$ be a decreasing sequence of closed sets generating $F$, i.e.,
\[ F = \{ \mathcal{C}(\mathbb{R}^{n}) : P_{n} \in F \text{ for some } n \} , \]
and suppose to the contrary that $\cap F_{n} \neq \emptyset$. It suffices to find two disjoint closed sets $H_{0}$ and $H_{1}$ such that $\forall n \in \mathbb{N} \exists x \in F_{n} \cap H_{0}$ and $x \in F_{n} \cap H_{1}$. For then if $U_{0}$ and $U_{1}$ are disjoint open neighborhoods of $H_{0}$ and $H_{1}$, one of $\mathbb{R}^{n} - U_{0}$ and $\mathbb{R}^{n} - U_{1}$ must belong to $F$, say $\mathbb{R}^{n} - U_{0}$, so $\mathbb{R}^{n} - U_{0} = \emptyset$ for some $n$, contradicting $H_{0} \cap F_{n} = \emptyset$.

In fact, we will construct discrete countable sets $H_{0}, H_{1}$ by induction. Write $B_{n} \in \mathbb{R}^{n}$ for the closed $n$-ball of radius $n$. Note that for each $n, F_{n} = \mathbb{B}_{n} \neq \emptyset$, since if $F_{n} \subseteq \mathbb{B}_{n}$ then $\forall k \in \mathbb{N} \exists f_{k} \in F_{n}$ so by compactness of $\mathbb{B}_{n}, \cap F_{n} = \emptyset$. Since $(F_{n})_{n}$ is decreasing, it follows that $F_{n} \subseteq F_{m}$ in is actually infinite. Now let $P_{0}, P_{1}$ be two distinct points in $\mathbb{R}^{n} - (B_{n} \setminus (k,m,i = 0,1))$. Then $H_{0} = \{ p_{i}^{n} : i \in \mathbb{N} \}$ and $H_{1} = \{ p_{i}^{n} : i \in \mathbb{N} \}$ are disjoint discrete sets.

Lemma 3: See Tognon (1972), Ch. V. Let $X \in \mathbb{R}^{n}$ be closed and let $\mathbb{m}_{X}$ denote the ideal in $C^{0}(\mathbb{R}^{n})$ consisting of functions which are flat on $X$. Let $(f_{n}^{x})_{n} \in \mathcal{C}(\mathbb{R}^{n})$ be a sequence of smooth functions on $\mathbb{R}^{n}$, and suppose $f_{n} \in \mathbb{m}_{X}$ for each $n$. Then there exists a smooth $g \in \mathcal{C}^{0}(\mathbb{R}^{n})$ with $Z(g) = X$, $g \geq 0$, such that $f_{n} \leq g$ for each $n$.

Proof of the theorem. Let $P$ be a given prime ideal in $C^{0}(\mathbb{R}^{n})$, generated by $(1_{F}^{x})_{x \in \mathcal{X}}$. By Lemma 1 and 2, $Z(P) = \cap Z(f) = \{ 0 \}$. Now suppose $P$ is not maximal, i.e., $x_{i} \in P$ for some $i \in \{ 1, \ldots , n \}$ ($x_{i}$ denote the projection $\mathbb{R}^{n} \to \mathbb{R}$). We claim that one may assume that $x_{i} \in P$ for all $i = 1, \ldots , n$. Indeed, suppose some $x_{i} \in P$, say $x_{n} \in P$. Write
\[ C^{0}(\mathbb{R}^{n}) \xrightarrow{q_{n}} C^{0}(\mathbb{R}^{n}) \]
for the homomorphism induced by the projection $\mathbb{R}^{n} \to \mathbb{R}^{n-1}$. From the fact that $x_{n} \in P$ it follows that
\[ q_{n}^{-1}(P) = \{ f_{n}((x_{1}, \ldots , x_{n-1}, 0)) \} \subset \mathcal{X} . \]

Rings of Smooth Functions

i.e., $q_{n}^{-1}(P)$ is again countably generated, and $q_{n}^{-1}(P)$ cannot be maximal since $P$ isn't. Repeating this, we eliminate all the coordinates $x_{i} \in P$.

So assume $x_{i} \notin P$ for $i = 1, \ldots , n$. If $g \in P$, then $q(0) = 0$, so for each $i$ we can write $g(x) = x_{i}h_{i}(x)$ for some smooth $h_{i}$. Since $x_{i} \notin P$ we have $h_{i} \notin P$, so $0 = h(0) = \frac{\partial g}{\partial x_{i}}(0)$. Repeating this we find $P \subseteq \mathcal{m}_{1}$. By Lemma 3 above, there exists a smooth function $g \in \mathcal{m}_{1}$ with $g(0) = 0$, $Z(g) = \{ 0 \}$, such that $f_{k} \leq g \cdot h_{k}$ for each $k$, say $f_{k} = g \cdot h_{k}$ with $h_{k} \in \mathcal{m}_{1}$.

For $A \subseteq \{ 1, \ldots , n \}$, let
\[ Q_{A} = \{ q_{i}^{\mathcal{X}} \}_{i \in A} \text{ if } i \subset A, x_{i} \notin \text{ if } i \notin A \]
where $A^{c}$ is the complement of $A$. By the Whitney extension theorem (Malgrange (1966), Chapter I) or Tognon (1972), Chapter IV) there exists for each $A$ a smooth function $q_{A} : \mathbb{R}^{n} \to \mathbb{R}$ with $q_{A}|_{Q_{A}} = g|_{Q_{A}}, q_{A}|_{A^{c}} = 0$.

Since the $q_{A}$'s cover $\mathbb{R}^{n}$, the product of all the $q_{A}$'s is identically zero, so one of the $q_{A}$'s must be in $P$, say $q_{A} \in P$. Hence there are $q_{i}^{\mathcal{X}} \in C^{0}(\mathbb{R}^{n})$ ($i = 1, \ldots , l$) such that $q_{A}(x) = \prod_{1}^{l} q_{i}^{\mathcal{X}}(x) q_{i}^{\mathcal{X}}(x) = \prod_{1}^{l} q_{i}^{\mathcal{X}}(x) h_{i}(x) q_{i}(x) . \]

Thus for $x \in Q_{A} \cap \{ 0 \}$, $\prod_{1}^{l} q_{i}^{\mathcal{X}}(x) h_{i}(x) = 1$, contradicting $h_{i} \in \mathcal{m}_{1}$.

This proves the theorem.

For a $C^{0}$-ring $A$, we define $\text{Spec}(A)$ to be the set of $C^{0}$-radicals, prime ideals in $A$, equipped with the Zariski topology defined by the basic opens $\text{B}(a) = \{ P | a \notin P \}$, for $a \in A$.

Let us give a more explicit description of the spaces that arise as spectra of $C^{0}$-rings. If $A = C^{0}(\mathbb{R})/I$, and $I = (f)(f(x))$, then $\text{Spec}(A)$ is the space of prime filters in the lattice $\mathcal{F}$ of zero sets in $\mathbb{R}$ (see part I, §1) which contain $\mathcal{I}$, having as a base for the topology the sets
\[ Q_{P} = \{ P | f \notin P \} \]
for each zero set $f$. ($Q_{P}$ corresponding to $B(f)$ if $P = B(f)$). Alternatively, let $\mathcal{B}$ be the lattice of open subsets of $\mathbb{R}$ which are complements of zero sets. Then prime filters $P$ in $\mathcal{B}$ correspond to prime filters $Q$ in $\mathcal{B}$, by
\[ P \mapsto (P \setminus B(\mathcal{U}^{P})), Q \mapsto (Q \cup B(\mathcal{U}^{Q})) \]
and this bijection maps prime filters $P \ni \Phi$ to prime filters $Q$ which are proper on $\Phi$ in the sense that

$$V \ni Q \forall P \ni \Phi \cup P = \Phi \ .$$

So $\text{Spec}(A)$ is the space of prime filters of opens in $\Phi$ which are proper on "the underlying space" $\Phi$, with the Stone topology having as basic opens the sets

$$B(U) = \{P|U \ni P\} \ , \ U \subset \Phi .$$

1.4. Lemma. Let $A$ be a $C^n$-ring.

(i) $B(a) \sqsubseteq B(b)$ iff $b$ is invertible in $A[a^{-1}]$, iff $a \in \sqrt{b}$.

(ii) Each basic open $B(a)$ of $\text{Spec}(A)$ is compact; in fact $B(a) \in \bigcup_i B(a_i)$

iff there are finitely many indices $1, \ldots, n$ such that

$$B(a) \subseteq B(a^n_1 \cdots a^n_n) \ .$$

(iii) The basic opens $B(a)$ form a distributive lattice, with

$$B(a) \cap B(b) = B(ab), \quad B(a) \cup B(b) = B(a^n b^n) .$$

Proof. Let $A = C^n(\mathbb{R}^n)/\mathbb{Z}$, and $\Phi = \{Z(f)|f \in I\}$ the corresponding filter in $\Phi$, as above. To prove (i), let $a, b$ be represented by functions $a(x), b(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ (depending only on a finite set $E$ of coordinates). Then $B(a) \sqsubseteq B(b)$ iff for every prime filter $P$ in $\Phi$, $P \ni a \Rightarrow Z(a) \subseteq Z(b)$.

This is equivalent to $\mathbb{R}^n \ni P \ni Z(b)$, because $Z(a)$ is an arbitrary standard lattice theoretic argument. In other words, $a \in b$, but this means precisely that $b$ is invertible in $A[a^{-1}] = C^n(\mathbb{R}^n)/I[1/U]$. This proves (i).

To prove (ii), let $a, a_i$ be represented by functions $a(x), a_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$. Then $B(a) \subseteq B(a_i)$ for every prime filter $P$ in $\Phi$ we have that $P \ni a \Rightarrow Z(a) \subseteq Z(a_i)$.

This is equivalent to $a^n_1 \cdots a^n_n$ being invertible in $A[a^{-1}]$, thus proving (ii).

(iii) is now clear.

Recall from Hochster (1969) that a topological space is spectral if it has a basis of compact opens which is closed under finite intersections, and each irreducible closed subset of the space has a unique generic point.

1.5. Corollary. $\text{Spec}(A)$ is a spectral space.

To give a simple example, if $M \subset \mathbb{R}^n$ is a manifold, the spectrum of $C(M)$ is the space of prime filters of open subsets of $M$ with the Stone topology, which is precisely the reflection of $M$ into the category of spectral spaces.

Rings of Smooth Functions

We may define a sheaf $\check{A}$ of $C^n$-rings on the space $\text{Spec}(A)$ by setting for basic opens $B(a)$

$$\check{A}(B(a)) = A[a^{-1}] .$$

If $B(a) \sqsubseteq B(b)$ there is a canonical restriction map $\check{A}(B(b)) \rightarrow \check{A}(B(a))$ by 1.4 (i). To show that this indeed defines a sheaf on $\text{Spec}(A)$, it suffices by lemma 1.4 to show that for $a, b \in A$, the square of canonical maps

$$A((a^n b^n)^{-1}) \longrightarrow A(a^{-1})$$

$$\downarrow \downarrow$$

$$A(b^{-1}) \longrightarrow A((a \cdot b)^{-1})$$

is a pullback. Since $A$ is a filtered colimit of finitely generated $C^n$-rings, it suffices to prove that such a square is pullback in the case that $A$ is finitely generated, say $A = C^\infty(\mathbb{R}^n)/I$. In other words, we have to show that for open

$$U, V \subset \mathbb{R}^n,$$

$$C^n(\mathbb{R}^n)/I[1/U] \longrightarrow C^n(\mathbb{R}^n)/[1/V]$$

$$\downarrow \downarrow$$

$$C^n(\mathbb{R}^n)/[1/V] \longrightarrow C^n(\mathbb{R}^n)/[1/U]$$

is a pullback. So assume $f : U \rightarrow \mathbb{R}$ and $g : V \rightarrow \mathbb{R}$ agree on $U \cap V$ modulo $I[1/U]$, i.e. there are $h_i \in I$ and $\phi_i \in C^n(\mathbb{R}^n)$ with $f - g = h_i \phi_i$. Let $\rho_U, \rho_V$ be a partition of unity subordinate to the cover $U, V$ of $U \cup V$, and set $h = \rho_U f + \rho_V g \in C^n(U \cap V)$. Then $\rho_U, \phi_i \in C^n(U)$, and for $x \in U$ we have

$$h(x) = \rho_U(x)f(x) + \rho_V(x)g(x)$$

$$= \rho_U(x)\phi_i(x)f(x) + h_i \phi_i(x)\phi_i(x)$$

$$= (\rho_U(x)\phi_i(x)f(x) - h_i \phi_i(x)\phi_i(x)) + h_i \phi_i(x)\phi_i(x) ,$$

so $h | U = f | U$ mod$I[1/U]$. Similarly $h | V = g | V$ mod$I[1/V]$. Uniqueness of $h$ is also straightforward.

Thus we have proved that $\check{A}$ defines a sheaf. Note that the stalk of $\check{A}$ at a point $P \in \text{Spec}(A)$ is the $C^n$-ring $A_P$, which is a local ring by lemma 1.1. So we have

1.6. Proposition. $\check{A}$ is a sheaf of $C^n$-rings on the space $\text{Spec}(A)$, and all its stalks are local $C^n$-rings.
Rings of Smooth Functions

Proof: That \( A \xrightarrow{\eta} A_P \) satisfies condition (1) of 1.8 has been shown in lemma 1.1. Conditions (2) and (3) are clear from Part I, theorem 1.4.

For the converse, suppose we are given a localization \( A \xrightarrow{h} L \), and let \( m \) be the maximal ideal of \( L \). Then \( P = \varphi^{-1}(m) \) is a \( C^\infty \)-radical prime ideal in \( A \), and by definition of \( A_P \) there is a unique local \( C^\infty \)-homomorphism \( h \) making the following diagram commute:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & A_P \\
\downarrow{h} & & \downarrow{h} \\
L & = & L
\end{array}
\]

The following lemma now completes the proof.

1.10. Lemma. Suppose we are given a commutative diagram of \( C^\infty \)-rings and \( C^\infty \)-homomorphisms:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & A' \\
\downarrow{h} & & \downarrow{h'} \\
L & = & L'
\end{array}
\]

where \( \psi, \psi' \) are localization and \( h \) is local. Then \( h \) is an isomorphism.

Proof: \( h \) is surjective. Take \( \ell \in L' \), and write \( \psi'(a) = \psi(a) \cdot \ell \) with \( \psi'(a) \) invertible. Then \( \psi(a) \) is also invertible since \( h \) is local. So we may define \( \ell = \psi(a)^{-1} \psi'(a)^{-1} \ell \), and \( h(\ell) = \ell' \). \( h' \) is injective: If \( \ell \in L \) and \( h(\ell) = 0 \), write \( L \cdot \psi'(a) = \psi(a) \) with \( \psi(a) \) invertible (1.8.2), so \( \psi(a) = 0 \) and hence (1.8.3) there is a \( \ell \in L \) with \( \psi(a) \cdot \psi(a) = 0 \) and \( \psi'(a) \) invertible, or equivalently since \( h \) is local, \( \psi(a) \) invertible. Then \( 0 = \psi(a) \cdot \ell \cdot \psi(a) \), and \( \phi(a) \cdot \psi(\ell) = 0 \).

By proposition 1.9, the maps \( A = A_P \), \( P \) a \( C^\infty \)-radical prime ideal in \( A \), are precisely the localizations of \( A \). This is not to say that if \( P \) is a prime ideal in \( A \) such that \( h_P \) is local, \( P \) must be \( C^\infty \)-radical, as the following example shows.

1.11. Example. Note that theorem 1.4 of Part I easily implies that if \( A \) is a \( C^\infty \)-domain and \( a \in A \), then \( A[a^{-1}] \) is again a \( C^\infty \)-domain (provided it is non-trivial, i.e. \( a \in \varphi'(0) \)). In particular, if \( N \) is a manifold, \( U \subset N \) is open, and \( x \) is a prime ideal in \( C^\infty(M) \), then \( (1)U \) is a prime ideal in \( C^\infty(U) \). So if \( P \) is a prime ideal in \( C^\infty(M) \) with \( I \subset P \),

\[
(C^\infty(M)/I)_P = \lim_{I \subset P} C^\infty(U_i)/U_i
\]
Rings of Smooth Functions

1.14. Theorem. Let $M$ be a manifold, and $P$ a prime ideal in $C^\infty(M)$. Then $C^\infty(M)_P$ is a local $C^\bullet$-ring iff $P$ is $C^\bullet$-radical.

Proof: $\Leftarrow$ is Lemma 1.1. For $x \in P$ suppose $C^\infty(M)_P \subseteq \lim_{\to} C^\bullet(X_U^p)$ is a local ring, so we have a localization $\eta : C^\infty(M) \to C^\infty(M)_P$. Let $m$ be the maximal ideal of $C^\infty(M)_P$, and let $I = \eta^{-1}(m)$, which is a $C^\bullet$-radical prime ideal in $C^\infty(M)$. It is easy to see that $I$ is isomorphic to $\cap_{n=1}^{\infty} \eta^{-1}(m)$.

Let $\mathcal{P}$ be the filters of closed sets corresponding to $I$ and $P$ respectively, as in part (1). By Lemma 2.1 of part (1), both $\mathcal{P}$ and $\mathcal{P}$ are prime filters, and $\mathcal{P} \subseteq \mathcal{P}$ since $I \subseteq P$. Moreover, if $P \subseteq M$ is closed and $U \subseteq M$ is open, then it follows from (1) that

$R \in \mathcal{P}$ if and only if $R \in \mathcal{P}$.

(2)

It now suffices to show that $\mathcal{P} = \mathcal{P}$. For this implies that $\eta \in \mathcal{P} = \mathcal{P}$, hence $P = I$, so $P$ is $C^\bullet$-radical. To see this, choose a closed $X \subseteq M$ with $x \in X$, say $x \in Z(g)$ for some $g \in P$, and assume to the contrary that $x \notin \mathcal{P}$. Let

$h = e^{-1/2}f$, $t = e^{-1/h^2}$.

Then $Z(g) = Z(f) = Z(h) = X$, and both $h$ and $f$ are well-defined $C^\bullet$-functions, so $h \in \mathcal{P}$ and $f \in \mathcal{P}$.

From the fact that $\mathcal{P}$ is a prime filter, it follows immediately that for every $F \subseteq \mathcal{P}$ there is a (countable) family of pairwise disjoint compact subsets $F_i \in \mathcal{P}$ such that $U^c \subseteq \mathcal{P}$.

Therefore we may without loss assume that $X$ is the disjoint union of compact sets $X_1^+$, and by choosing the $X_1^+$ small enough, $X^c \subseteq \mathcal{P}$ where the $U_i$ are coordinate neighbourhoods. Fix $x_i \in U_i^c$ where $U_i \cap M = \emptyset$ and identify $U_i^c$ with $M_m = \dim M$.

Let

$$b_i = (ux_i^1|d(x_i,k_i)^{1/2})$$

$$a_i = (ux_i^{k_i-1}|d(x_i,k_i)^{1/2})$$

By (2) we have that $\bigcup_{i=1}^{\infty} \mathcal{P}_i \subseteq \mathcal{P}$, so writing $b_i = \sum_{n=0}^{\infty} b_n$ and $a_i = \sum_{n=0}^{\infty} a_n$, we find that either $U_j \subseteq \bigcup_{i=1}^{\infty} \mathcal{P}_i$ or $U_j \subseteq \bigcup_{i=1}^{\infty} \mathcal{P}_i$, since $X$ is a prime filter.

Let us say $U_j \subseteq \bigcup_{i=1}^{\infty} \mathcal{P}_i$.

Define a function $k_i : N \to N$ for each $i$ by

$k_i(n) = \sup_{x_n^k} h(x) \leq \frac{1}{k_i(n)}$.
Rings of Smooth Functions

So the function \( u^t : \mathbb{R}^t \rightarrow \mathbb{R} \)

\[
u^t(x) = \sum_{P \in \mathcal{P}_{2n}} \varphi_P(x) \cdot f(x)
\]

is well-defined, and vanishes on \( x^t \). We claim that \( u^t \) is smooth. (At points outside \( x^t \) there clearly is no problem.) Indeed, by the chain rule we have

\[
|D^m\varphi_P(x)| \leq (2^{k^t(2n)}) |a| \cdot K_a
\]

for some constant \( k_a \) depending only on \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m \). But \( k^t(2n) \leq (\text{sup}(h(x)|x^t|^{1/2}))^{-1} \), so

\[
|D^m\varphi_P(x)| \leq K_a \cdot 2 |a| \cdot |D^m f(x)| / h(x)
\]

Since \( \frac{D^m f(x)}{h(x)|\alpha|} \) converges to 0 as \( x \) converges to \( x \) by definition of \( f \) and \( h \), we conclude that \( u^t \) is smooth.

Since the \( u^t \) have disjoint closures, we can fix a smooth function \( u : \mathbb{R}^m \rightarrow \mathbb{R} \) with \( u|_{(1)^t} = u^t \). Then \( u \), \( f \) on \( \cup_{n \in \mathbb{N}} \mathbb{R}^n \), \( e \), so \( u \in I \) because \( f \in I \) and \( I \) is \( C^m \)-radical.

On the other hand, we claim that for any \( v \in \mathbb{C}(\mathbb{R}) \)

(3)

\[
2(v) = 2(u) \Rightarrow v \in P
\]

which by (1) would imply \( u \in I \), a contradiction which would complete the proof. It thus remains to show (3).

So take such \( v \). It suffices to prove that \( v/u \) is a smooth function, since \( u \in P \). We work on each \( u^t \) separately. As with the smoothness of \( u^t \), the only problem is at points in \( x^t \in \mathbb{R}^t \), i.e. we have to show that for a sequence \( \{x_p\} \) in \( \mathbb{R}^t \),

\[
\frac{D^m v(x_p)}{g^m(x_p)} \rightarrow 0 \quad \text{if } x_p \rightarrow x \in \mathbb{R}^t
\]

for each \( \alpha = (\alpha_1, \ldots, \alpha_m) \), \( \alpha > 0 \). Let

\[
\alpha = \sup \{|D^m v(x)| |D^m f(x)| x^t \}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_m) \). By the mean value theorem

\[
|D^m v(x_p)| \leq \alpha \cdot (1+c) \cdot (k^t(2n))^{-1}
\]

where \( k^t(2n) \) is the ring to which \( x_p \) is close. So \( n \rightarrow \infty \) as \( p \rightarrow \infty \). Thus
2. The spectrum of a \( C^m \)-ringed topos

We will now define the spectrum of a \( C^m \)-ring in an arbitrary topos. In the case of the topos \( \text{Sets} \), this coincides with the spectrum as defined in the preceding section. As pointed out in the introduction, the construction of the spectrum should be considered as the construction of the universal (the "best", the free) localization of a \( C^m \)-ring, and this point of view can only be rigorously pursued if one works in a topos theoretic context. It will be shown that this more general notion of spectrum to be described below has the right universal properties, and in particular a strengthening of theorem 1.7 will be proved. As such, all this is just an instance of the general theory of spectra originating with Hakim (1972). For a perspicuous exposition of this general theory we refer to Chapter 6 of Johnstone (1977); see also the references cited there.

Needless to say, by a topos we will always mean a Grothendieck topos. (We will in general be rather sloppy about 2-categorical details concerning geometric morphisms, and just stick in "up to isomorphism" now and then. These details, however, are completely straightforward, and the reader can certainly supply them if he feels the need to do so.)

Recall from part I that a \( C^m \)-ring in a topos \( E \) is a functor from the category \( C^m \) (the category of Euclidean spaces \( \mathbb{R}^n \), \( n \geq 0 \), and smooth maps) into \( E \) which preserves finite products. Localizations of \( C^m \)-rings in a topos \( E \) are defined as before, but using the internal logic of \( E \). Thus

2.1. Definition. Let \( A \) be a \( C^m \)-ring in a topos \( E \). A \( C^m \)-homomorphism \( A \to L \) in \( E \) is called a localization of \( A \) if conditions (1), (2), (3) of 1.8 are valid in \( E \).
Proof. (In the back of our minds, we will have sheaf semantics for a site of defi-
nition for \( \mathcal{E} \).) It suffices to prove the lemma for the case that \( \mathcal{A} \) is finitely generated, and (by passing to an appropriate stage in the site) that the generators are global sections of \( \mathcal{A} \). Let \( \Gamma \) be the global sections functor and the constant sheaf functor. Then the free \( C^\infty \)-ring on \( n \) generators in \( \mathcal{E} \) is \( \Delta(C^\infty(\mathcal{U}^\iota)) \), since \( C^\infty(\mathcal{U}^\iota) \) is free on \( n \) generators in \( \mathcal{S} \) and \( \Delta \) is left adjoint to \( \Gamma \). So \( \mathcal{A} \) is of the form \( \Delta(C^\infty(\mathcal{W}^1))/I \) for some ideal \( I \) in \( \mathcal{E} \) (a subsheaf of \( \Delta(C^\infty(\mathcal{W}^1)) \)). Let \( a \in \mathcal{A} \), again without loss a global section. By passing to an appropriate cover in the
site, we may assume that \( a \) is represented by a constant element, i.e., there is an \( f \in C^\infty(\mathcal{U}^\iota) \) such that \( a = \Delta(f) \mod I \).

To prove properties (1) and (2), it suffices to take \( I = (0) \), since inverting \( a \) and quotienting by \( I \) is interchangeable. But in \( \mathcal{E} \),
\[
\Delta(C^\infty(\mathcal{W}^1))/I \cong \Delta(C^\infty(\mathcal{W}^1)(I^{-1})) \]
\[
= \Delta(C^\infty(\mathcal{U}^\iota)),
\]

again since \( \Delta \dashv \Gamma \), and properties (1) and (2) now follow straightforwardly from the corresponding properties for \( C^\infty(\mathcal{U}^\iota) \) in \( \mathcal{S} \).

(N.B. The quantifiers in (1) and (2) should be interpreted in \( \mathcal{E} \), i.e., as prescribed by sheaf semantics for \( \mathcal{E} \).)

2.3. Lemma. Suppose we are given a commutative diagram
\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\
\downarrow{h} & & \downarrow{h'} \\
\mathcal{L} & \xrightarrow{\psi} & \mathcal{U}
\end{array}
\]

of \( C^\infty \)-rings and \( C^\infty \)-homomorphisms in a topos \( \mathcal{E} \), where \( \varphi \) and \( \psi \) are local-
izations and \( h \) is local. Then \( h \) is a local isomorphism.

Proof. The proof for the case of \( \mathcal{S} \), i.e., lemma 1.10, is completely constructive, so can be performed within \( \mathcal{E} \). Alternatively, all the notions involved are coherent, so it suffices to prove the special case \( \mathcal{E} = \mathcal{S} \).

2.4. Lemma ("factorization lemma"). Let \( \mathcal{E} \) be a topos, and let \( A \xrightarrow{\psi} B \) a hom-
omorphism of \( C^\infty \)-rings, with \( B \) a local ring. Then there exists a unique (up to
isomorphism) factorization
\[
\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{f} & & \downarrow{\varphi} \\
\mathcal{L} & \xrightarrow{\psi'} & \mathcal{U}
\end{array}
\]
such that \( \varphi \) is a localization, and \( f \) is a local \( C^\infty \)-homomorphism.

Proof. (We work inside \( \mathcal{E} \).) Let \( \mathcal{E} = (\mathcal{A} \times \mathcal{B}; \mathcal{A} \times \mathcal{B} \text{ is invertible} \) \). We have a factorization
\[
\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{\varphi} & & \downarrow{\psi'} \\
\mathcal{L} & \xrightarrow{\psi''} & \mathcal{U}
\end{array}
\]

where \( \varphi \) is the canonical map. Moreover, since \( A(\mathcal{L}; \mathcal{E}) = \varlim 
\mathcal{E}^\infty(\mathcal{L}; \mathcal{E}) \), it follows from lemma 2.2 that conditions 2.1 (2) and 2.1 (3) are satisfied by \( \varphi \).

So it remains to show that \( A(\mathcal{L}; \mathcal{E}) \) is local. Take \( a_1, a_2 \in A(\mathcal{L}; \mathcal{E}) \) with \( a_1 + a_2 \) invertible. By 2.2 we can write \( a_1 \cdot \psi(c_1) = \varphi(d_1) \) with \( c_1, d_1 \in A \) and \( \psi(c_1) \) invertible. Then \( \varphi(a_1 a_2) \psi(c_1 c_2) = \varphi(d_1 c_2 d_2 c_1) \in \mathcal{U}(\mathcal{E}) \), so \( d_1 \in \mathcal{E} \) or \( d_2 \in \mathcal{E} \) since \( B \) is a local ring, so \( a_1 \) is invertible or \( a_2 \) is.

Uniqueness of the factorization \( \varphi = \varphi \psi \) follows immediately from lemma 2.3 and the universal property of \( \varphi \).

Notice that by uniqueness, the factorization of lemma 2.4 is preserved by inverse images of geometric morphisms, since the defining property is a coherent one.

We are now ready to show that the spectrum as defined in the beginning of this section has the required universal property. The category of \( C^\infty \)-ringed toposes has as objects pairs \( (E, A) \), a \( C^\infty \)-ring in the topos \( E \), while a morphism \( (F, B) \to (E, A) \) consists of a geometric morphism \( \mathcal{E} \to \mathcal{F} \) together with a \( C^\infty \)-homomorphism \( \mathcal{E}(A) \to \mathcal{B} \) in \( \mathcal{F} \). The category of locally \( C^\infty \)-ringed toposes is the subcategory having as objects pairs \( (E, A) \) with \( A \) a local \( C^\infty \)-ring in \( E \), and as morphisms such pairs \( (F, B) \) with \( \psi \) a local \( C^\infty \)-homomorphism in \( \mathcal{F} \).

2.5. Theorem. \( \text{Spec} \) defines a functor from \( C^\infty \)-ringed toposes to locally \( C^\infty \)-ringed toposes which is right adjoint to the inclusion functor.

Proof. Let \( (F, B) \to (E, A) \) be a morphism of \( C^\infty \)-ringed toposes, with \( B \) a local ring in \( \mathcal{F} \). We need to show that there is a unique geometric morphism \( \mathcal{E} \to \mathcal{F} \)
and a unique local \( \mathbb{C} \)-homomorphism \( \hat{E}(A) \to B \) making the appropriate triangles commute. Since the construction of \( \hat{E}(A) \) is preserved by pullback, we may assume \( F = E, \ f = \text{id} \). By lemma 2.4, the \( \mathbb{C} \)-homomorphism \( A \to B \) corresponds to a localization \( A \to L \) of \( A \), and since by definition \( \hat{E}(A) \) classifies localizations of \( A \), we obtain a unique geometric morphism \( g \),

\[
\begin{array}{ccc}
\hat{E} & \overset{g}{\to} & E \\
\downarrow & & \downarrow \\
\text{Spec}(A) & & \text{Spec}(A)
\end{array}
\]

with \( g^*(p^*(A) \to A) \cong A \to L \).

We will now give a more concrete description of the spectrum of a \( \mathbb{C} \)-ringed topos \( (E, A) \), which will at the same time show that the geometric morphism \( \hat{E}(A) \to E \), the counit of the adjunction of theorem 2.5, is localic. Moreover, it will then become clear that the construction of the spectrum as described in section 1 is a special case (viz. where \( E \) = Sets) of the spectrum of a \( \mathbb{C} \)-ringed topos as defined above.

So let \( (E, A) \) be a \( \mathbb{C} \)-ringed topos. We define a distributive lattice \( \text{D}^m(A) \) in \( E \) as follows. For each \( a \in A \) we introduce a symbol \( \text{D}^m(a) \). The \( \text{D}^m(a) \)'s are preordered by

\[
\text{D}^m(a) \leq \text{D}^m(b) \text{ iff } b \text{ is invertible in } A[a^{-1}].
\]

It is convenient to identify \( \text{D}^m(a) \) and \( \text{D}^m(b) \) if \( \text{D}^m(a) \cong \text{D}^m(b) \cong \text{D}^m(a) \), i.e. if \( A[a^{-1}] \cong A[b^{-1}] \), so as to obtain a poset which we denote by \( \text{D}^m(A) \). To show that \( \text{D}^m(A) \) is a distributive lattice, we need parts (a) and (b) of the following lemma. (c) will be used later on.

2.6. Lemma. Let \( A \) be a \( \mathbb{C} \)-ring in a topos \( E \). Then in \( E \) it holds that for any sequence \( a_1, \ldots, a_n \) of elements of \( A \), and any \( b \in B \), (a) \( \text{D}^m(a_1^2 + \ldots + a_n^2) \text{ is invertible in each of } A[a_i^{-1}] (i = 1, \ldots, n) \)

(b) \( b \text{ is invertible in } A[a_1^{-1}], \ldots, A[a_n^{-1}] \) if, then also in \( A[a_1^{-2}], \ldots, a_n^{-2}] \)

(c) \( a_1 + \ldots + a_n \text{ is invertible in } A \), then so is \( a_1^2 + \ldots + a_n^2 \).

Proof. (a) It suffices to prove this for finitely generated \( A \), and we can assume that \( A = \Delta(C(a^m)) \) as in the proof of lemma 2.2. Also (by passing to an appropriate cover in the site, etc.), we may without loss assume that the \( a_i \) are constant, i.e. in the image of \( \Lambda \). But then, just as in the proof of 2.2, we have reduced to proving (a) for finitely generated \( C \)-rings in Sets. This is trivial: let \( A = \mathbb{C}(B)/I \), \( a_i \) represented by \( a_i(x) : x^m + W \). Using part I, proposition 1.2, it suffices to observe that \( \bigcup_{a_i \in U_{a_i}} \subseteq U_{a_1^2 \ldots a_n^2} \).

(b) As in (a), we need only prove this for finitely generated \( C \)-rings \( A = \mathbb{C}(B)/I \) in Sets, which is again easy, using that for \( f, g \in \mathbb{C}(B) \), \( g \) is invertible in \( A[e^{-1}] \) iff \( \exists h \in I \text{ s.t. } h \subseteq U_{a_i} \).

(c) The assertions are coherent, so it suffices to prove (c) for the case \( E = \text{Sets} \). This is easy and left to the reader.

2.7. Corollary. \( \text{D}^m(A) \) is a distributive lattice, with

\[
\begin{align*}
\text{D}^m(a_1)^{\ldots} \text{D}^m(a_n) &= \text{D}^m(a_1 \ldots a_n) \, , \\
\text{D}^m(a_1)^{\ldots} \text{D}^m(a_n) &= \text{D}^m(a_1^2 \ldots a_n^2) \, .
\end{align*}
\]

Let \( \mathbb{C}(\Delta(D^m(A))) \) be the topos of sheaves in \( E \) on \( D^m(A) \) (regarded as a site in \( E \) with the finite cover topology), or equivalently, the topos of sheaves in \( E \) on the locale \( \text{Id}_{D^m(A)} \) of ideals in \( D^m(A) \), and let \( \gamma : \mathbb{C}(\Delta(D^m(A))) \to E \) be the canonical localic geometric morphism.

2.8. Proposition. The functor \( \bar{A} : D^m(a) \mapsto A(a^{-1}) \) defines a sheaf on \( D^m(A) \) in \( E \), and the canonical map \( \gamma^*(A) \to \bar{A} \) is a localization of \( \gamma^*(A) \) in \( \mathbb{C}(\Delta(D^m(A))) \).

Proof. To show that \( \bar{A} \) is a sheaf, it suffices to show in \( E \) that diagrams of the form

\[
\begin{array}{ccc}
A(b_1 b_2^{-1}) & \to & A(b_1^{-1}) \\
\downarrow & & \downarrow \\
A(b_2^{-1}) & \to & A(b_1 b_2^{-1})
\end{array}
\]

are pullbacks. By arguing as in the proof of lemma 2.2, it is only necessary to show this for the case where \( E = \text{Sets} \). This has been shown already in section 1, see proposition 1.6.

Next, we show that \( \bar{A} \) is a local ring in \( \mathbb{C}(\Delta(D^m(A))) \). Still reasoning inside \( E \), suppose \( x_1, x_2 \in \bar{A}(D^m(A)) = A(a^{-1}) \) and \( D^m(A) \) \( \Gamma \)-reduces \( x_1 x_2 = 1 \) in \( A \) (here \( \Gamma \)-refers to forcing over \( D^m(A) \), in \( E \)). By lemma 2.2 it suffices to consider the special case where \( x_1, x_2 \in A \) (more precisely, \( x_1, x_2 \in A(D^m(A)) \) are restrictions...
of elements, also called \( x_1, x_2 \) of \( \tilde{\mathbb{A}}(\mathbb{D}'(\mathfrak{a})) = \mathfrak{A} \). \( \mathbb{D}'(\mathfrak{a}) \triangleright x_1 + x_2 = 1 \)
implies that \( x_1 + x_2 \) is invertible in \( A[\alpha^{-1}] \), hence by 2.6 (c) \( x_1^2 + x_2^2 \) is
invertible in \( A[\alpha^{-1}] \), and therefore \( \mathbb{D}'(\mathfrak{a}) \triangleright x_1 \mathbb{D}'(\mathfrak{a}) + \mathbb{D}'(\mathfrak{a}) \). Thus
\( \mathbb{D}'(\mathfrak{a}) \triangleright x_1 \mathbb{D}'(\mathfrak{a}) \triangleright x_2 \mathbb{D}'(\mathfrak{a}) \).

Finally, the fact that \( \gamma^*(\mathfrak{A}) \triangleright \mathbb{A} \) satisfies conditions (2) and (3) of localization
(definition 2.1) in \( \mathbb{E}(\mathbb{D}'(\mathfrak{A})) \) follows trivially from 2.2, since the map \( \gamma^*(\mathfrak{A}) \triangleright \mathbb{A} \)
as a natural transformation in \( \mathbb{E} \) has as component over \( \mathbb{D}'(\mathfrak{a}) \) the canonical
\( \mathbb{C} \)-homomorphism \( \mathfrak{A} \triangleright A[\alpha^{-1}] \).

2.9. Lemma. The construction of the locally \( \mathbb{C} \)-ringed topos \( (\mathbb{E}(\mathbb{D}'(\mathfrak{A})), \mathbb{A}) \)
is pullback stable, i.e., if \( F \longrightarrow E \) is a geometric morphism, we have a pullback square

\[
\begin{array}{ccc}
F(\mathbb{D}'(\mathfrak{a})) & \longrightarrow & \gamma^*F \\
\downarrow f^* & & \downarrow f \\
\mathbb{E}(\mathbb{D}'(\mathfrak{A})) & \longrightarrow & E
\end{array}
\]

and \( f^*(\mathfrak{A}) \triangleright f^*(\mathbb{A}) \).

Proof. This is obvious, once it has been observed that the construction of the
universal map \( \mathfrak{A} \triangleright A[\alpha^{-1}] \) is stable, i.e., \( f^*(\mathfrak{A} \triangleright A[\alpha^{-1}]) = f^*(\mathfrak{A}) \triangleright f^*(A)(f^*(\alpha)^{-1}) \).

2.10. Theorem. The topos \( \mathbb{S}\mathbb{E}_{\mathbb{C}}(\mathfrak{A}) \) as defined in the beginning of this section
is canonically equivalent (over \( \mathbb{E} \)) to the topos \( \mathbb{E}(\mathbb{D}'(\mathfrak{A})) \), and this equivalence
identifies the \( \mathbb{A} \)'s:

\[
(\mathbb{E}(\mathbb{D}'(\mathfrak{A})), \mathbb{A}) \cong (\mathbb{S}\mathbb{E}_{\mathbb{C}}(\mathfrak{A}), \mathbb{A}) \text{ over } \mathbb{E}.
\]

Proof. We will show that \( (\mathbb{E}(\mathbb{D}'(\mathfrak{A})), \mathbb{A}) \) also classifies localizations of \( \mathfrak{A} \).
That is, for any topos \( F \longrightarrow E \) over \( \mathbb{E} \), a localization \( f^*(\mathfrak{A}) \triangleright L \) in \( F \)
corresponds to a geometric morphism \( F \).

\[
\begin{array}{ccc}
F & \longrightarrow & \mathbb{E}(\mathbb{D}'(\mathfrak{A})) \\
\downarrow f & & \downarrow f \\
E & \longrightarrow & \mathbb{E}(\mathfrak{A})
\end{array}
\]

with \( \mathbb{E}(\mathfrak{A}) \triangleright L \) of 2.8 to \( f^*(\mathfrak{A}) \triangleright L \). By lemma 2.9,
we have to consider the special case where \( f = \text{id} \), \( F = E \), i.e., we have to show in \( E \) that localizations of \( \mathfrak{A} \) correspond to points of the site \( \mathbb{D}'(\mathfrak{A}) \), i.e., to prime filters in \( \mathbb{D}'(\mathfrak{A}) \).

Rings of Smooth Functions and Their Localization, II

In one direction, given a localization \( \mathfrak{L} \longrightarrow L \), let \( P = \{ \mathfrak{a}(a) \mid \psi(a) \text{ is invertible} \} \). Clearly, \( P \) is a filter. It is also prime, since \( \mathfrak{L} \) is local.

Conversely, given a prime filter \( P \) in \( \mathbb{D}'(\mathfrak{A}) \), i.e., an \( \mathbb{E}(\mathfrak{A}) \)-point \( P \longrightarrow \mathbb{E}(\mathbb{D}'(\mathfrak{A})) \), \( \mathfrak{L} + P \mathfrak{a}(a) \triangleright \text{lim}_{\mathfrak{L} \rightarrow \mathfrak{A}(a^{-1})} \mathfrak{A}(a^{-1}) \) is a localization of \( \mathfrak{A} \), being the pullback along \( P \)
of the localization \( \gamma^*(\mathfrak{A}) \triangleright \mathbb{A} \) of 2.8.

These processes are inverse to each other: the way round, starting with \( \mathfrak{L} \longrightarrow L \), \( \mathfrak{L} + P \mathfrak{a}(a) \) gives an isomorphic localization, by 2.3 and (the proof of) 2.4. And the other way round, starting with \( P \) and defining a prime filter

\[
P' = \{ \mathfrak{a}(a) \mid U(\mathbb{D}'(\mathfrak{a}))(\mathfrak{a}(a)^{-1}) \}
\]
clearly gives \( P \) back, i.e., \( P = P' \).

2.11. Remark. A third way of constructing \( \mathbb{S}\mathbb{E}_{\mathbb{C}}(\mathfrak{A}) \) is by adapting Joyal's notion of
universal support to the case of \( \mathbb{C} \)-rings. For a presentation of the usual spectrum of commutative ring theory along those lines see Wraith (1979).

Given the explicit description of \( \mathbb{S}\mathbb{E}_{\mathbb{C}}(\mathfrak{A}) \) as sheaves on \( \mathbb{D}'(\mathfrak{A}) \) (theorem 2.10),
it follows immediately from lemma 1.4 that for a \( \mathbb{C} \)-ring \( \mathfrak{A} \) in \( \text{Sets} \), \( \mathbb{S}\mathbb{E}_{\mathbb{C}}(\mathbb{A}, \mathbb{A}) \)
as defined in section 1 coincides with \( \mathbb{S}\mathbb{E}_{\mathbb{C}}(\mathbb{A}, \mathbb{A}) \) as defined in the
beginning of this section. Consequently, the adjunction of theorem 1.7 becomes a special case
of theorem 2.5.

3. The archimedean spectrum of a \( \mathbb{C} \)-ring

In Dubuc (1981), a construction of a spectrum of \( \mathbb{C} \)-rings (in \( \text{Sets} \)) is given,
which is adjoint to the global functor, but not on the category of \( \mathbb{C} \)-spaces as in
our theorem 1.7; instead, Dubuc restricts his attention to the full subcategory of those
\( \mathbb{C} \)-spaces all whose stalks are Archimedean local \( \mathbb{C} \)-rings. (Warning:
Dubuc uses "local" for "Archimedean and local"!) We will now show how Dubuc's
adjunction can be obtained from ours, simply by composing the adjunction of theorem
1.7 with the coreflection into \( \mathbb{C} \)-spaces with Archimedean stalks.

3.1. Lemma. Let \( \mathfrak{A} \) be a local \( \mathbb{C} \)-ring, say \( \mathfrak{A} \cong \mathbb{C}[\mathfrak{a}]^\mathfrak{a} / I \).
The following are equivalent:

(i) \( \mathfrak{A} \) is Archimedean,

(ii) \( \mathbb{C}(\mathfrak{a}) = \mathfrak{A}(\mathfrak{a}) / I \) is non-empty,

(iii) the residue field of \( \mathfrak{A} \) is isomorphic to \( \mathfrak{R} \).

Proof. (i) \( \Rightarrow \) (ii) Archimedeaness of \( \mathfrak{A} \) refers to the canonical preorder \( \leq \) on
\( \mathfrak{A} \) (see part I, end of section I). We first consider the case where \( \mathfrak{a} \) is
finitely generated, i.e., \( \mathfrak{A} \) is finite, say \( \mathfrak{A} \cong \mathbb{C}[\mathfrak{a}]^\mathfrak{a} / I \). For each \( i = 1, \ldots, n \), we have
for some \( k_1 \in \mathbb{N} \) that \( -k_1 < s_1 < k_1 \) in \( A \). So if \( f \in \mathcal{I} \), \( X \in \mathcal{Z}(f) \), then \( f(z) < s_1 \). Let \( k = \max(k_1, \ldots, k_n) \), \( f = \frac{r_1^2 + \cdots + r_n^2}{t^2} \). Then \( Z(f) \subset [-k, k]^n \), i.e., the filter \( \{ Z(f) : f \in \mathcal{I} \} \) contains a compact element. Therefore \( \cap \{ Z(f) : f \in \mathcal{I} \} = \emptyset \).

In the general case, write \( A = \varprojlim A_D / D \) ranging over finite subsets of \( E \), where \( A_D = C(\mathbb{R})^{/D}_D \), \( I_D = \{ f \in \mathbb{R} : f\chi_D \} \). This \( I_D \) is the \( \pi \)-adic completion of \( A_D \). Now, all we claim that each \( A_D \) is Archimedean. To see this, let \( g \in C(\mathbb{R})^E_D \) be an element of \( A_D \). Since \( g \notin A \) for some \( n \in \mathbb{N} \), there is a finite \( \mathbb{Z}' = 0 \) and an \( f \in I_D \) such that \( Z(f) = \{ x \in \mathbb{R} : g(x) = 0 \} \). Write \( h = \text{h}(x) \in \mathbb{R} \) being a smooth function with \( Z(h) = \{ x \in \mathbb{R} : g(x) = 0 \} \). Then \( h \notin I_D \), hence \( g \notin A_D \), and hence \( g \notin A \).

By the argument for finitely generated \( A \) (which did not use the assumption that \( A \) is a local ring), we conclude that for each finite \( D \subset E \), \( Z(I_D) = \emptyset \), in fact that \( \mathcal{I} = I \). But this easily implies that \( \mathcal{I} = \emptyset \). Then \( I_D = I_D \), and \( Z(I_D) = \emptyset \). Since \( Z(I_D) \) is a prime ideal, it is trivial that \( Z(I_D) \) is a prime ideal. Since for \( D' \supset D, Z(I_D) \subset Z(I_{D'}) \subset \mathbb{R}^{-1}D \), we determine a unique point \( x \in \mathbb{R}^{-1}D \), and \( x \in Z(I_D) \).

(iii) \( \Rightarrow \) (i). If \( x \in Z(I) \), then the unique maximal ideal in \( \mathbb{C}(\mathbb{R})^E \) containing \( x \) must be \( \{ f(x) = 0 \} \), so this implication is clear.

(iii) \( \Rightarrow \) (i). The projection \( A \to \mathbb{R} \) is a local \( \mathbb{C} \)-homomorphism, so it reflects the order \( < \). Thus \( A \) is Archimedean since \( \mathbb{R} \) is.

Now consider the full subcategory of \( \mathbb{C}(\mathbb{R})^E \) defined in section 1, consisting of objects \( (x, \mathcal{O}_x^E) \) such that all the stalks \( \mathcal{O}_{x,y} \) are Archimedean local \( C \)-rings. We call such \( \mathbb{C} \)-spaces \( \mathbb{C} \)-Archimedean. Let

\[ i : \text{Archimedean \( \mathbb{C} \)-spaces} \to \text{\( \mathbb{C} \)-spaces} \]

be the inclusion functor. \( i \) has a right adjoint \( \text{Ar} \), defined as follows. For a \( \mathbb{C} \)-space \( (x, \mathcal{O}_x^E) \), \( \text{Ar}(x, \mathcal{O}_x^E) \) is the pair \( (x, \mathcal{O}_x) \), where \( y = (x, \mathcal{O}_x, X, X, X, \ldots) \) is Archimedean, and \( \mathcal{O}_x \) is the sheaf on \( y \) having the same stalks as \( \mathcal{O}_x \). \( \text{Ar} \) is a functor, as follows immediately from the fact that if \( A \to B \) is a local \( \mathbb{C} \)-homomorphism of local \( C \)-rings and \( B \) is Archimedean, then so is \( A \). For the same reason, \( \text{Ar} \) is right adjoint to \( i \), for \( (x, \mathcal{O}_x^E) \) in \( \mathbb{C} \)-spaces, \( (x, \mathcal{O}_x) \) is Archimedean, so \( \text{Ar}(x, \mathcal{O}_x^E) = (x, \mathcal{O}_x) \) is Archimedean since \( \mathcal{O}_{x,y}^E \) is.

When we compose this adjunction \( i \to \text{Ar} \) with the adjunction

\[ \mathcal{I}(\mathbb{C} \text{-rings}) \to \text{\( \mathbb{C} \)-spaces} \]

of theorem 1.7, we obtain the following theorem, which is theorem 9 of Dubuc (1981). So Dubuc's Spec is our \( \text{Ar} \to \text{Spec} \).

3.2. Theorem. The \text{global sections} functor

\[ \Gamma : \text{Archimedean \( \mathbb{C} \)-spaces} \to \text{\( \mathbb{C} \)-rings} \]

has a right adjoint \( \text{Ar} \to \text{Spec} \), the "Archimedean spectrum".

It is easy to compute \( \text{Ar} \to \text{Spec} \) explicitly. If \( A = C(\mathbb{R})^E \), then the points of \( \text{Ar} \to \text{Spec} \) are the \( \mathbb{C} \)-radical prime ideals \( P \supset I \) such that \( A_P \) is Archimedean. Then we have a \( \mathbb{C} \)-homomorphism \( A \to A_P \). It is clear that the \( \mathbb{C} \)-radical prime ideals \( P \) correspond to the evaluation at a point \( x \in Z(I) \), and since \( A \to \mathbb{R} \) is local, we find by lemma 1.1 that \( P = \{ x \in \mathbb{R} : g(x) = 0 \} \), i.e., \( P \) is maximal. So we have the following corollary.

3.3. Corollary. Let \( A = C^E \). Then \( \text{Ar} \to \text{Spec}(A) \) is the space \( Z(I) \) with the subspace topology inherited from the product topology on \( \mathbb{R}^E \).

Dubuc also observes that the functor \( \text{Ar} \to \text{Spec} \) of theorem 3.2 is full and faithful when restricted to finitely generated \( \mathbb{C} \)-rings which are what he calls "of local character" (this is also often called "germ-determined"), see Dubuc (1981), theorem 11.

Finally, let us briefly indicate how to generalize the spectrum à la Dubuc for arbitrary \( \mathbb{C} \)-ringed toposes.

One approach would be to add one more axiom to the definition of a localization \( A \to A' \) of a \( \mathbb{C} \)-ring \( A \) in a topos \( E \) (see 2.1), namely

\[ \forall \ell \in E \bigvee_{\ell < n} \ell \in \mathbb{C}(\mathbb{R})^E \]

(4)

This axiom is geometric, so we may construct the classifying topos for the generic Archimedean localization, parallel to the first construction of the spectrum in section 2. We write \( \text{Ar} \to \text{Spec}(E, A) \) for the spectrum of \( (E, A) \) defined in this way. So \( \text{Ar} \to \text{Spec}(E, A) \) is an Archimedean local \( \mathbb{C} \)-ringed topos.

There is also a second, equivalent approach, which shows that this Archimedean spectrum is localic over \( E \), and specializes to the construction described above in the case where \( E = \text{Sets} \). Given a \( \mathbb{C} \)-ring \( A \) in \( E \), one defines a locale \( \mathbb{C}(\mathbb{R})^E(A) \) in \( E \) by taking the same poset \( \mathbb{C}(\mathbb{R})^E(A) \) (as in section 2), and equipping it with a covering system generated by the following covers:

(1) the empty family covers \( \mathbb{C}(\mathbb{R})^E(0) \).
(ii) \( D^\omega(a_1, \ldots, a_n) \) is covered by \( \{ D^\omega(a_1), \ldots, D^\omega(a_n) \} \).

(iii) for each \( a, b \in A \), \( D^\omega(a) \) is covered by

\[ D^\omega(\chi(\lambda-b)) \mid \lambda \in \mathbb{R} \],

here \( \mathbb{E} \) is the object of integers in \( E \), and \( \chi \) is the interpretation in \( A \) of a non-negative characteristic function \( \mathbb{E} \to \mathbb{R} \) for the positive reals. Write \( \mathbb{E}[D^\omega_A(\lambda)] \) for the topos of sheaves in \( E \) on this locale. \( D^\omega_A(\lambda) \) is a sublocale of \( D^\omega(A) \) (regarded as a locale, rather than a lattice), so we have an inclusion of toposes

\[ \mathbb{E}[D^\omega_A(\lambda)] \hookrightarrow \mathbb{E}[D^\omega(A)]. \]

The covers of type (iii) precisely force \( i^*(\lambda) \) to be Archimedean in \( \mathbb{E}[D^\omega_A(\lambda)] \). Parallel to the arguments in section 2, one can show that the Archimedean locally \( C^\infty \)-ringed topos \( (\mathbb{E}[D^\omega_A(\lambda)], i^*(\lambda)) \) classifies Archimedean localizations of \( A \).

So it must coincide with \( A \) Spec(\( E, A \)). Moreover, for the case where \( E = \text{Sets} \), it coincides with Dubuc's spectrum.

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