LECTURE 1: HOMOTOPY AND THE FUNDAMENTAL GROUPOID

You probably know the fundamental group \( \pi_1(X,x_0) \) of a space \( X \) with a base point \( x_0 \), defined as the group of homotopy classes of maps \( S^1 \to X \) sending a chosen base point on the circle to \( x_0 \). In this course we will define and study higher homotopy groups \( \pi_n(X,x_0) \) by using maps \( S^n \to X \) from the \( n \)-dimensional sphere to \( X \), and see what they tell us about the space \( X \).

Let us begin by fixing some preliminary conventions. By 

**space** we will always mean a Hausdorff topological space (later, we may narrow this down to various other important classes of spaces, such as compact spaces, locally compact spaces, compactly generated spaces, or CW-complexes). A 

**map** between two such spaces will always mean a continuous map.

A **pointed space** is a pair \((X,x_0)\) consisting of a space \( X \) and a base point \( x_0 \in X \). A **map** between pointed spaces \( f: (X,x_0) \to (Y,y_0) \) is a map \( f: X \to Y \) with \( f(x_0) = y_0 \). A **pair of spaces** is a pair \((X,A)\) consisting of a space \( X \) and a subspace \( A \subseteq X \). A **map of pairs** \( f: (X,A) \to (Y,B) \) is a map \( f: X \to Y \) such that \( f(A) \subseteq B \).

1. Homotopy of maps

We now recall the central notion of a homotopy.

**Definition 1.1.** Two maps of spaces \( f, g: X \to Y \) are called homotopic if there is a continuous map \( H: X \times [0,1] \to Y \) such that:

\[
H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x), \quad x \in X
\]

Such a map \( H \) is called a homotopy from \( f \) to \( g \). We write \( f \simeq g \) to denote such a situation or \( H: f \simeq g \) if we want to be more precise.

The homotopy relation enjoys the following nice properties.

**Proposition 1.2.** Let \( X, Y, \) and \( Z \) be spaces.

(i) The homotopy relation is an equivalence relation on the set of all maps from \( X \) to \( Y \).

(ii) If \( f, g: X \to Y \) and \( k, l: Y \to Z \) are homotopic then also \( kf, lg: X \to Z \) are homotopic.

**Proof.** Let us prove the first claim. So, let us consider maps \( f, g, h: X \to Y \). The homotopy relation is reflexive since we have the following constant homotopy at \( f \):

\[
\kappa_f: f \simeq f \quad \text{given by} \quad \kappa_f(x,t) = f(x)
\]

Given a homotopy \( H: f \simeq g \) then we obtain an inverse homotopy \( H^{-1} \) as follows:

\[
H^{-1}: g \simeq f \quad \text{with} \quad H^{-1}(x,t) = H(x,1-t)
\]

Thus, the homotopy relation is symmetric. Finally, if we have homotopies \( F: f \simeq g \) and \( G: g \simeq h \) then we obtain a homotopy \( H: f \simeq h \) by the following formula:

\[
H(x,t) = \begin{cases} 
F(x,2t) & , \quad 0 \leq t \leq 1/2 \\
G(x,2t-1) & , \quad 1/2 \leq t \leq 1
\end{cases}
\]

Thus we showed that the homotopy relation is an equivalence relation.
To prove the second claim let us begin by two special cases. Let us assume that we have a homotopy \( H : f \simeq g \). Then we obtain a homotopy from \( kf \) to \( kg \) simply by post-composition with \( g \):

\[
k H : X \times [0,1] \xrightarrow{H} Y \xrightarrow{k} Z
\]
is a homotopy \( kH : kf \simeq kg \).

The next case is slightly more tricky. Given a homotopy \( K : k \simeq l \) then we obtain a homotopy from \( kg \) to \( lg \) as follows:

\[
K \circ (g \times \text{id}) : X \times [0,1] \xrightarrow{} Y \times [0,1] \xrightarrow{} Z
\]
defines a homotopy \( K \circ (g \times \text{id}) : kg \simeq lg \).

In order to obtain the general case it suffices now to use the transitivity of the homotopy relation since from the above two special cases we deduce \( kf \simeq kg \simeq lg \) as intended. This concludes the proof.

The equivalence classes with respect to the homotopy relation are called homotopy classes and will be denoted by \([f]\). Given two spaces \( X \) and \( Y \) then the set of homotopy classes of maps from \( X \) to \( Y \) is denoted by \([X,Y]\).

This proposition allows us to form a new category where the objects are given by spaces and where morphisms are given by homotopy classes of maps. Let us begin by recalling the notion of a category.

**Definition 1.3.** A category \( \mathcal{C} \) consists of the following data:

(i) A collection \( \text{ob}(\mathcal{C}) \) of objects in \( \mathcal{C} \).

(ii) Given two objects \( X, Y \) there is a set \( \mathcal{C}(X,Y) \) of morphisms in \( \mathcal{C} \).

(iii) Associated to three objects \( X, Y, Z \) there is a composition map:

\[
\circ : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \rightarrow \mathcal{C}(X,Z) : (g,f) \mapsto g \circ f
\]

This datum has to satisfy the following two properties:

- The composition is associative, i.e., we have \((h \circ g) \circ f = h \circ (g \circ f)\) whenever these expressions make sense.
- For every object \( X \) there is an identity morphism \( \text{id}_X \in \mathcal{C}(X,X) \) such that for all morphisms \( f \in \mathcal{C}(X,Y) \) we have:

\[
id_Y \circ f = f = f \circ \text{id}_X
\]

We use the following standard notation. Given a category \( \mathcal{C} \) we write \( X \in \mathcal{C} \) in order to say that \( X \) is an object of \( \mathcal{C} \). If \( f \in \mathcal{C}(X,Y) \) is a morphism from \( X \) to \( Y \) we write \( f : X \rightarrow Y \). Moreover, the composition is often just denoted by juxtaposition, i.e., we write \( gf \) instead of \( g \circ f \).

You know already a lot of examples of categories.

**Example 1.4.**

(i) The category of sets and maps of sets.

(ii) The category of groups with group homomorphisms.

(iii) Given a ring \( R \) we have the category of \( R \)-modules and \( R \)-linear maps.

(iv) The category of fields and field extensions.

(v) The category of smooth manifolds and differentiable maps.

Using the conventions introduced above we also have the following examples.

**Example 1.5.**

(i) The category \( \text{Top} \) of spaces and maps.

(ii) The category \( \text{Top}_* \) of pointed spaces and maps of pointed spaces.
(iii) The category $\text{Top}^2$ of pairs of spaces and maps of pairs.

Now, as a consequence of the above proposition we can form a new category with objects given by spaces and maps given by homotopy classes of maps. Two homotopy classes can be composed by forming the composition of representatives and then passing to the corresponding homotopy class. The above proposition guarantees that this is well-defined. It is easy to check that we get a category this way.

**Corollary 1.6.** Spaces and homotopy classes of maps define a category $\text{Ho}(\text{Top})$, the (naive) homotopy category of spaces.

There are variants of this for the case of $\text{Top}_*$ and $\text{Top}^2$. In these two cases we are mainly interested in slightly different variants of the homotopy relation.

**Definition 1.7.** Two maps $f, g: (X, x_0) \to (Y, y_0)$ in $\text{Top}_*$ are called homotopic relative to base points, notation

$$f \simeq g \text{ rel } x_0,$$

if there exists a homotopy $H: f \simeq g$ such that

$$H(x_0, t) = y_0, \quad t \in [0, 1].$$

Thus, we are asking for the existence of a homotopy through pointed maps. Again, one checks that this is an equivalence relation which is compatible with composition. The equivalences classes $[f]$ are called pointed homotopy classes and the set of all such is denoted by $[(X, x_0), (Y, y_0)]$.

These pointed variants are in fact special cases of the following more general notion.

**Definition 1.8.** Let $(X, A)$ be a pair of spaces, $Y$ a space, and $f, g: X \to Y$ two maps such that $f(a) = g(a)$ for all $a \in A$. Then a homotopy from $f$ to $g$ relative to $A$ is a homotopy $H: f \simeq g$ such that $H(a, -): [0, 1] \to Y$ is constant for all $a \in A$. Thus the additional condition imposed is

$$H(a, t) = f(a) = g(a), \quad t \in [0, 1], \quad a \in A.$$

If for two such maps $f$ and $g$ there is a relative homotopy $H$, then this will be denoted by

$$H: f \simeq g \text{ rel } A.$$

This notion specializes to pointed homotopy or homotopy, if $A$ consists of a single point or is empty respectively.

**Definition 1.9.** Let us consider two maps $f, g: (X, A) \to (Y, B)$ in $\text{Top}^2$. A homotopy of pairs from $f$ to $g$ is a homotopy $H: f \simeq g$ which, in addition, satisfies

$$H(a, t) \in B, \quad t \in [0, 1], \quad a \in A.$$

This is of course precisely the condition that for each $t \in [0, 1]$ the map $H(-, t): X \to Y$ is actually a map of pairs $(X, A) \to (Y, B)$.

Also in this case we obtain a well-behaved equivalence relation and the equivalence classes are denoted as before. We will write $[(X, A), (Y, B)]$ for the set of homotopy classes of pairs. As a consequence of this discussion we obtain the following result.

**Corollary 1.10.**

(i) Pointed spaces and pointed homotopy classes define a category $\text{Ho}(\text{Top}_*)$, the homotopy category of pointed spaces.
(ii) Pairs of spaces and relative homotopy classes define a category $\text{Ho(Top)}^2$, the homotopy category of pairs of spaces.

**Definition 1.11.** A map $f : X \to Y$ of spaces is a **homotopy equivalence** if there is a map $g : Y \to X$ such that

$$g \circ f \simeq \text{id}_X \quad \text{and} \quad f \circ g \simeq \text{id}_Y.$$  

The space $X$ is **homotopy equivalent** to $Y$, notation $X \simeq Y$, if there is a homotopy equivalence $f : X \to Y$.

It is easy to see that being homotopy equivalent is an equivalence relation. The equivalence class of a space $X$ with respect to this relation is called the **homotopy type** of $X$.

**Definition 1.12.** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an **isomorphism** if there is a morphism $g : Y \to X$ such that

$$g \circ f = \text{id}_X \quad \text{and} \quad f \circ g = \text{id}_Y.$$  

An object $X$ is **isomorphic** to $Y$ if there is an isomorphism $X \to Y$.

**Example 1.13.**

(i) A morphism in the category of sets is an isomorphism if and only if it is bijective.

(ii) In many categories the objects are given by ‘sets with additional structure’ while morphisms are defined as morphisms of sets ‘respecting this additional structure’. Frequently, it is true that a morphism in such a category is an isomorphism if and only if the underlying map of sets is a bijection. This is for example the case for groups, rings, fields, and modules.

(iii) In the category $\text{Top}$ one has to be careful; a continuous bijection $f : X \to Y$ is, in general, not an isomorphism, i.e., a homeomorphism. For this to be true we have to impose additional conditions on the spaces (like compact and Hausdorff).

(iv) A morphism $[f] : X \to Y$ in $\text{Ho(Top)}$ is an isomorphism if and only if any map $f : X \to Y$ representing this homotopy class is a homotopy equivalence.

**Exercise 1.14.**

(i) Define a notion of **pointed homotopy equivalence** (without using the concept of an isomorphism) and check that the pointed analog of Example 1.13(iv) holds.

(ii) Define a notion of **relative homotopy equivalence** (again, without using the concept of an isomorphism) and check that the relative analog of Example 1.13(iv) holds.

(iii) The notion of being isomorphic is an equivalence relation on the collection of objects in an arbitrary category. The corresponding equivalence classes are called isomorphism classes.

2. **The fundamental groupoid**

Recall that given a space $X$ and an element $x_0 \in X$ we have the fundamental group $\pi_1(X, x_0)$ of homotopy classes of loops at $x_0$. If we take a different point $x_1 \in X$ we obtain a further such group $\pi_1(X, x_1)$ which a priori has nothing to do with $\pi_1(X, x_0)$. However, if $x_0$ and $x_1$ lie in the same path component of $X$ then any path joining them induces an isomorphism between the two homotopy groups by conjugation with the given path. It is easy to check that paths homotopic to the boundary induce the same isomorphism. However, in general, the induced isomorphisms may be different. A convenient way of encoding all these different groups and isomorphisms in a single structure is given by the **fundamental groupoid** of a space. In order to discuss this we have to introduce one more notion from category theory.

**Definition 1.15.** A category $\mathcal{C}$ is a **groupoid** if every morphism in $\mathcal{C}$ is an isomorphism.
This terminology reflects the idea that a groupoid is like a group in a certain sense. In fact, for every object in a groupoid the set of endomorphisms is actually a group. The justification for this terminology is given by the first of the following examples.

**Example 1.16.**

(i) Every group $G$ gives rise to a groupoid $BG$ as follows. The category $BG$ has a single object denoted by $\ast$. Hence, the only set of morphisms we have to specify is the set of endomorphisms of $\ast$ and this set is given by:

$$BG(\ast, \ast) = G.$$  

The composition is given by the multiplication of the group. It is easy to check that all the axioms of a groupoid are precisely fulfilled because $G$ is a group. In other words, a group is essentially the same thing as a groupoid with one object. Similarly, a monoid $M$ is essentially the same thing as a category with a single object.

(ii) Every category has an underlying groupoid given by the same objects and the isomorphisms only. For example, we have the category of sets and bijections, spaces and homeomorphisms, smooth manifolds and diffeomorphisms, and so on.

We now give a description of the fundamental groupoid $\pi(X)$ of a space $X$. The collection of objects $\text{ob}(\pi(X))$ is given by the points of $X$. Given two points $x, y \in X$ the set of morphisms

$$\pi(X)(x, y)$$

is given by the set of homotopy classes of paths from $x$ to $y$ relative to the boundary. To be completely specific, let us recall that a path $\alpha$ in $X$ from $x$ to $y$ is given by a map

$$\alpha: [0, 1] \to X$$

such that $\alpha(0) = x$ and $\alpha(1) = y$.

We want to consider two such paths $\alpha$ and $\beta$ to be equivalent if they are homotopic relative to the boundary, i.e., if there is a map $H: [0, 1] \times [0, 1] \to X$ such that

$$H(0, -) = \alpha, \quad H(1, -) = \beta, \quad H(t, 0) = x, \quad \text{and} \quad H(t, 1) = y \quad \text{for all} \quad 0 \leq t \leq 1.$$  

Now, given a path $\alpha$ from $x$ to $y$ and a path $\beta$ from $y$ to $z$ then the concatenation $\beta * \alpha$ is given by:

$$(\beta * \alpha)(t) = \begin{cases} 
\alpha(2t), & 0 \leq t \leq 1/2 \\
\beta(2t - 1), & 1/2 \leq t \leq 1 
\end{cases}$$

**Exercise 1.17.**

(i) The concatenation $\beta * \alpha$ is again continuous and defines a path from $x$ to $z$.

(ii) Given three points $x, y, z \in X$ then the assignment $([\beta], [\alpha]) \mapsto [\beta * \alpha]$ defines a well-defined composition map

$$\circ: \pi(X)(y, z) \times \pi(X)(x, y) \to \pi(X)(x, z).$$

(iii) The composition is associative.

(iv) Prove that the homotopy classes of constant paths give identity morphisms. Thus we already know that $\pi(X)$ defines a category.

(v) Given a path $\alpha$ from $x$ to $y$ then we the inverse path $\alpha^{-1}$ from $y$ to $x$ is defined by the formula

$$\alpha^{-1}(t) = \alpha(1 - t).$$

Show that every morphism $[\alpha]$ in $\pi(X)$ is an isomorphism by verifying that $[\alpha^{-1}]$ defines a two-sided inverse of $[\alpha]$. 


This exercise shows that \( \pi(X) \) is a groupoid, the \textit{fundamental groupoid} of \( X \). Let \( (X, x_0) \) be a pointed space, then the \textit{fundamental group} \( \pi_1(X, x_0) \) of \( X \) at \( x_0 \) is defined as the group of automorphisms of \( x_0 \) in \( \pi(X) \), i.e.,

\[
\pi_1(X, x_0) = \pi(X)(x_0, x_0)
\]

Let us introduce the following notation for the unit interval, its boundary, and the sphere:

\[
I = [0, 1], \quad \partial I = \{0, 1\}, \quad \text{and} \quad S^1 = \{x \in \mathbb{R}^2 \mid ||x|| = 1\}
\]

Then, the fundamental group is given by

\[
\pi_1(X, x_0) = [(I, \partial I), (X, x_0)] \cong [(S^1, *), (X, x_0)]
\]

where the latter isomorphism comes from the homeomorphism \( I/\partial I \cong S^1 \).

**Example 1.18.** We assume you have learned about the fundamental group in your undergraduate topology, and know examples like

\[
\pi_1(S^1, *) \cong \mathbb{Z} \quad \text{and} \quad \pi_1(T, *) \cong \mathbb{Z} \times \mathbb{Z},
\]

where

\[
T = S^1 \times S^1
\]

is the torus. The latter formula is of course a special case of the following \textit{product formula}. Given \( (X, x_0), (Y, y_0) \in \textbf{Top}_* \), then the product \( (X \times Y, (x_0, y_0)) \) is again a pointed space and the natural map

\[
\pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)
\]

is an isomorphism.
LECTURE 2: SPACES OF MAPS, LOOP SPACES
AND REDUCED SUSPENSION

In this section we will give the important constructions of loop spaces and reduced suspensions associated to pointed spaces. For this purpose there will be a short digression on spaces of maps between (pointed) spaces and the relevant topologies.

To be a bit more specific, one aim is to see that given a pointed space \((X,x_0)\), then there is an entire pointed space of loops in \(X\). In order to obtain such a loop space

\[ \Omega(X, x_0) \in \text{Top} \]

we have to specify an underlying set, choose a base point, and construct a topology on it. The underlying set of \(\Omega(X, x_0)\) is just given by the set of maps

\[ \text{Top}((S^1, *), (X, x_0)) \]

A base point is also easily found by considering the constant loop \(\kappa_{x_0}\) at \(x_0\) defined by:

\[ \kappa_{x_0} : (S^1, *) \to (X, x_0) : t \mapsto x_0 \]

The topology which we will consider on this set is a special case of the so-called compact-open topology. We begin by introducing this topology in a more general context.

1. Function spaces

Let \(K\) be a compact Hausdorff space, and let \(X\) be an arbitrary space. The set \(\text{Top}(K, X)\) of continuous maps \(K \to X\) carries a natural topology, called the compact-open topology. It has a subbasis formed by the sets of the form

\[ B(T, U) = \{ f : K \to X \mid f(T) \subseteq U \} \]

where \(T \subseteq K\) is compact and \(U \subseteq X\) is open. Thus, for a map \(f : K \to X\), one can form a typical basis open neighborhood by choosing compact subsets \(T_1, \ldots, T_n \subseteq K\) and small open sets \(U_i \subseteq X\) with \(f(T_i) \subseteq U_i\) to get a neighborhood \(O_f\) of \(f\),

\[ O_f = B(T_1, U_1) \cap \ldots \cap B(T_n, U_n) \cdot \]

One can even choose the \(T_i\) to cover \(K\), so as to ‘control’ the behavior of functions \(g \in O_f\) on all of \(K\).

The topological space given by this compact-open topology on \(\text{Top}(K, X)\) will be denoted by:

\[ X^K \in \text{Top} \]

**Proposition 2.1.** Let \(K\) be a compact Hausdorff space. Then for any \(X, Y \in \text{Top}\), there is a bijective correspondence between maps

\[ Y \xrightarrow{f} X^K \quad \text{and} \quad Y \times K \xrightarrow{g} X. \]
Proof. Ignoring continuity for the moment, there is an obvious bijective correspondence between such functions $f$ and $g$, given by

$$f(y)(k) = g(y, k)$$

for all $y \in Y$ and $k \in K$. We thus have to show that if $f$ and $g$ correspond to each other in this sense, then $f$ is continuous if and only if $g$ is.

In one direction, suppose $g$ is continuous, and choose an arbitrary subbasic open $B(T, U) \subseteq X^K$. To prove that $f^{-1}(B(T, U))$ is open, choose $y \in f^{-1}(B(T, U))$, so $g(\{y \times T\}) \subseteq U$. Since $T$ is compact and $g$ is continuous, there are open $V \supseteq y$ and $W \supseteq T$ with $g(V \times W) \subseteq U$ (you should check this for yourself!). Then surely $V$ is a neighborhood of $y$ with $f(V) \subseteq B(T, U)$, showing that $f^{-1}(B(T, U))$ is open.

Conversely, suppose $f$ is continuous, and take $U$ open in $X$. To prove that $g^{-1}(U)$ is open, choose $(y, k) \in g^{-1}(U)$, i.e., $y \in f^{-1}(B(\{k\}, U))$. Now, in the space $X^K$, if a function $K \to X$ maps $k$ into $U$, then it must map a neighborhood $W_k$ of $k$ into $U$, and if we choose $W_k$ small enough it will even map the compact set $T = W_k$ into $U$. This shows that:

$$B(\{k\}, U) = \bigcup\{B(T, U) \mid T \text{ is a compact neighborhood of } k\}$$

So $y \in f^{-1}(B(T, U))$ for some $T = W_k$. By continuity of $f$, we find a neighborhood $V$ of $y$ with $f(V) \subseteq B(T, U)$, i.e., $g(V \times T) \subseteq U$; and hence surely $g(V \times W_k) \subseteq U$, showing that $g^{-1}(U)$ is open.

As a special case we can consider the compact space $K = S^1$, for which maps out of $K$ are loops.

**Definition 2.2.** Let $X \in \mathbf{Top}$ and let $(Y, y_0) \in \mathbf{Top}_*$. 

1. The space $\Lambda(X) = X^{S^1} \in \mathbf{Top}$ is the **free loop space** of $X$.
2. The **loop space** $\Omega(Y, y_0) \in \mathbf{Top}_*$ of $(Y, y_0)$ is the pair consisting of the subspace of $\Lambda(Y)$ given by the pointed loops $(S^1, \ast) \to (Y, y_0)$ together with the constant loop $\kappa_{y_0}$ at $y_0$ as base point.

2. The reduced suspension

The above proposition applied to the compact space $K = S^1$ tells us that there is a bijective correspondence between maps $g: X \times S^1 \to Y$ and maps $f: X \to Y^{S^1} = \Lambda(Y)$. Let now $(X, x_0)$ and $(Y, y_0)$ be pointed spaces. We want to make explicit the conditions to be imposed on a map $g: X \times S^1 \to Y$ such that the corresponding map $f$ actually defines a pointed map:

$$f: (X, x_0) \to \Omega(Y, y_0) = (\Omega(Y, y_0), \kappa_{y_0})$$

Since the correspondence is given by the formula

$$g(x, t) = f(x)(t)$$

it is easy to check that these conditions are:

$$g(x_0, t) = y_0, \quad t \in S^1, \quad \text{and} \quad g(x, \ast) = y_0, \quad x \in X$$

Thus the map $g$ has to send the subspace $\{x_0\} \times S^1 \cup X \times \{\ast\} \subseteq X \times S^1$ to the base point $y_0 \in Y$ and hence factors over the corresponding quotient.

**Definition 2.3.** Let $(X, x_0)$ be a pointed space. Then the **reduced suspension** $\Sigma(X, x_0) \in \mathbf{Top}_*$ is the pointed space

$$\Sigma(X, x_0) = X \times S^1/\{ (x_0) \times S^1 \cup X \times \{\ast\} \}$$

where the base point is given by the collapsed subspace.
Using the quotient map $I \to I/\partial I \cong S^1$, a different description of the reduced suspension $\Sigma(X,x_0)$ is given by

$$\Sigma(X,x_0) = X \times I/(\{x_0\} \times I \cup X \times \partial I)$$

and in either description we will denote the base point by $\ast$. The quotient map $I \to I/\partial I \cong S^1$ induces maps of pairs

$$(X \times I, \{x_0\} \times I \cup X \times \partial I) \to (X \times S^1, \{x_0\} \times S^1 \cup X \times \{\ast\}) \to (\Sigma(X,x_0), \ast).$$

At the level of elements we allow us to commit a minor abuse of notation and simply write

$$(x,t) \mapsto [x,t], \quad x \in X, \quad t \in I.$$ 

Thus, we have the following descriptions of the base point $\ast \in \Sigma(X,x_0)$

$$[x_0,t] = [x,0] = [x,1] = \ast, \quad x \in X, \quad t \in I,$$

and similarly if the second factor is an element of $S^1$. Proposition 2.1 combined with the discussion preceding the definition of the reduced suspension gives us the following corollary.

**Corollary 2.4.** Let $(X,x_0), (Y,y_0) \in \text{Top}_\ast$. Then there is a bijective correspondence between pointed maps

$$g : \Sigma(X,x_0) \to (Y,y_0) \quad \text{and} \quad f : (X,x_0) \to \Omega(Y,y_0)$$

given by the formula $g([x,t]) = f(x)(t)$ for all $x \in X$ and $t \in S^1$.

This corollary turns out to be the special case of a pointed analog of Proposition 2.1. In order to understand this, we have to introduce a few constructions of pointed spaces. A pointed analog of spaces of maps is easily obtained (compare to the difference between the loop space and the free loop space).

**Definition 2.5.** Let $(K,k_0), (X,x_0) \in \text{Top}_\ast$ and assume that $K$ is compact. The **pointed mapping space**

$$(X,x_0)^{(K,k_0)} \subset X^K$$

is the subspace of pointed maps $(K,k_0) \to (X,x_0)$. It has a natural base point given by the constant map $\kappa_{x_0}$ with value $x_0$, and hence defines an object

$$(X,x_0)^{(K,k_0)} \in \text{Top}_\ast.$$ 

3. **The wedge sum and the smash product**

From now on we begin to be a bit sloppy about the notation of base points. If we do not need a special notation for a base point of a pointed space we will simply drop it from notation. For example, we will write ‘Let $X$ be a pointed space’, the suspension $\Sigma(X,x_0)$ will be denoted by $\Sigma(X)$, and similarly. Also the pointed mapping space of the above definition will sometimes simply be denoted by $X^K$. Moreover, we will sometimes generically denote base points by $\ast$. Whenever this simplified notation results in a risk of ambiguity we will stick to the more precise one.

As a next step, let us consider a pair of spaces $(X,A)$. Then we can form the quotient space $X/A$ by dividing out the equivalence relation $\sim_A$ generated by

$$a \sim_A a', \quad a,a' \in A.$$ 

The quotient space $X/A$ is naturally a pointed space with base point given by the equivalence class of any $a \in A$. In the sequel this will always be the way in which we consider a quotient space as a pointed space. Note that this was already done in the above definition of the (reduced) suspension of a pointed space.
Definition 2.6. Let \((X, x_0)\) and \((Y, y_0)\) be two pointed spaces, then their wedge \(X \vee Y\) is the pointed space
\[X \vee Y = X \sqcup Y/\{x_0, y_0\} \in \text{Top}_*.
\]
The wedge \(X \vee Y\) comes naturally with pointed maps \(i_X : X \to X \vee Y\) and \(i_Y : Y \to X \vee Y\).
In fact, this is the universal example of two such maps with a common target in the sense of the following exercise.

Exercise 2.7. Let \(X, Y,\) and \(W\) be pointed spaces and let \(f : X \to W, g : Y \to W\) be pointed maps.
Then there is a unique pointed map \((f, g) : X \vee Y \to W\) such that:
\[(f, g) \circ i_X = f : X \to W \quad \text{and} \quad (f, g) \circ i_Y = g : Y \to W.
\]
Thus, from a more categorical perspective the wedge is the categorical coproduct in the category of pointed spaces.

Example 2.8.
(i) The quotient space \(I/\{0, 1/2, 1\}\) is homeomorphic to \(S^1 \vee S^1\).
(ii) For any pointed space \(X\) we have \(X \vee * \cong X \cong * \vee X\).
(iii) For two pointed spaces \(X\) and \(Y\) we have \(X \vee Y \cong Y \vee X\).

The wedge \(X \vee Y\) of two pointed spaces is naturally a subspace of \(X \times Y\). In fact, this inclusion can be obtained by applying the above exercise as follows. For pointed spaces (!), the product \((X \times Y, (x_0, y_0))\) comes naturally with an inclusion map of \((X, x_0)\) given by
\[(X, x_0) \to (X \times Y, (x_0, y_0)) : x \mapsto (x, y_0).
\]
There is a similar map \((Y, y_0) \to (X \times Y, (x_0, y_0))\). Thus we are in the situation of Exercise 2.7 and hence obtain a pointed map \(X \vee Y \to X \times Y\). This map can be checked to be the inclusion of a subspace. The corresponding quotient construction is so important that it deserves a special name.

Definition 2.9. Let \(X\) and \(Y\) be pointed spaces. Then the smash product \(X \wedge Y\) of \(X\) and \(Y\) is the pointed space
\[X \wedge Y = X \times Y / (X \vee Y) \in \text{Top}_*.
\]
As it is the case for every quotient space, the smash product \(X \wedge Y\) naturally comes with a quotient map
\[q : X \times Y \to X \wedge Y.
\]
We will use the following notation for points in \(X \wedge Y\):
\[[x, y] = q(x, y), \quad x \in X, \quad y \in Y.
\]
In the next example, we will use the 0-dimensional sphere or 0-sphere \(S^0\) which is the two-point space:
\[S^0 = \{-1, +1\} \subseteq [-1, +1].
\]
Let us agree that we consider \(S^0\) as a pointed space with \(-1\) as base point. Moreover, let \(I_+\) be the disjoint union of \(I = [0, 1]\) with a base point, i.e.,
\[I_+ = [0, 1] \sqcup \ast \in \text{Top}_*.
\]
with \(\ast\) as base point.

Example 2.10.
(i) For every \(X \in \text{Top}_*\) we have \(X \wedge S^1 \cong \Sigma(X)\).
(ii) For every $X \in \text{Top}_*$ we have a homeomorphism $X \wedge S^0 \cong X \cong S^0 \wedge X$.

(iii) For two pointed spaces $X$ and $Y$ we have $X \wedge Y \cong Y \wedge X$.

(iv) Let $X \in \text{Top}_*$. The reduced cylinder of $X$ is the smash product $X \wedge I_+$. Unraveling the definition of the smash product, we see that we have

$$X \wedge I_+ \cong X \times I / \{x_0\} \times I \in \text{Top}_*.$$

Thus, pointed maps out of $X \wedge I_+$ are precisely the pointed homotopies.

**Proposition 2.11.** Let $K, X,$ and $Y$ be pointed spaces and assume that $K$ is compact, Hausdorff. Then there is a bijective correspondence between pointed maps $g : X \wedge K \to Y$ and $f : X \to Y^K$ given by the formula $g([x,k]) = f(x)(k)$ for all $x \in X$ and $k \in K$.

Using the cylinder construction on spaces one can also establish the following result. A proof will be given in the exercises.

**Corollary 2.12.** Let $K, X,$ and $Y$ be pointed spaces and assume that $K$ is compact, Hausdorff. Then there is a bijection:

$$[X \wedge K, Y] \cong [X, Y^K]$$

**Exercise 2.13.**

(i) Give a proof of Proposition 2.11 using the corresponding result about (unpointed) spaces. Hint: compare the proof of the special case of $K = S^1$ and realize that the smash product is designed so that this proposition becomes true.

(ii) Give a proof of Corollary 2.12. There are some hints on how to attack this on the exercise sheet.

We will now recall the notion of path components which allows us to establish a relation between loop spaces and the fundamental group.

Let $X$ be a space and let $x_0, x_1 \in X$. We say that $x_0$ and $x_1$ are equivalent, notation $x_0 \simeq x_1$, if and only if there there is a path in $X$ connecting them. It is easy to check that this defines an equivalence relation on $X$ with equivalence classes the path components of $X$. We write $\pi_0(X)$ for the set of path components of $X$ and denote the path component of $x \in X$ by $[x]$. A point in $X$ can be identified with a map $* : \to X$ sending the unique point $*$ to $x$. Under this identification, the above equivalence relation becomes the homotopy relation on maps $* \to X$. Thus, we have a bijection:

$$\pi_0(X) \cong [*, X]$$

In the case of a pointed space the set of path components has a naturally distinguished element given by the path component of the base point. Thus, the set $\pi_0(X, x_0)$ of path components of a pointed space $(X, x_0)$ is a pointed set. Now, a point $x \in (X, x_0)$ can be identified with a pointed map $S^0 \to X$. In fact, a bijection is obtained by evaluating such a map on $1 \in S^0$ (which is not the base-point!). Moreover, it is easy to see that we have an isomorphism of pointed sets

$$\pi_0(X, x_0) \cong [(S^0, -1), (X, x_0)].$$

**Exercise 2.14.** Verify the details of the above discussion.

As a consequence of our work so far we obtain the following corollary.
Corollary 2.15. Let \((X, x_0)\) be a pointed space. Then there is a canonical bijection of pointed sets:
\[
\pi_1(X, x_0) \cong \pi_0(\Omega(X, x_0))
\]

Proof. It suffices to assemble our results from above. Since the sphere \(S^1\) is a compact, Hausdorff space we can apply Corollary 2.12 in order to obtain:
\[
\pi_0(\Omega(X, x_0)) \cong \left([S^0, -1], \Omega(X, x_0)\right) \cong \left([S^0, -1] \wedge (S^1, *), (X, x_0)\right) \cong \pi_1(X, x_0)
\]
The third identification is a special case of Example 2.10(ii).

Thus, at least for theoretical purposes, in order to calculate the fundamental group of a given space it is enough to calculate the path components of the associated loop space. However, this does not really simplify the task since the loop space is, in general, less tractable than the original space. This corollary also suggests a definition of higher homotopy groups. Namely, given a pointed space \((X, x_0)\) we could simply set
\[
\pi_n(X, x_0) = \pi_0((X, x_0)(S^n, *))\quad n \geq 2.
\]
This will be pursued further in the next section where we will see that we actually obtain abelian groups this way.

The final aim of this section is to introduce the notion of a functor and to remark that many of the constructions introduced so far are in fact functorial. Here is the key definition.

Definition 2.16. Let \(\mathcal{C}\) and \(\mathcal{D}\) be categories. A functor \(F: \mathcal{C} \to \mathcal{D}\) from \(\mathcal{C}\) to \(\mathcal{D}\) is given by:

(i) An object function which assigns to each object \(X \in \mathcal{C}\) an object \(F(X) \in \mathcal{D}\).

(ii) For each pair of objects \(X, Y \in \mathcal{C}\) a morphism function \(\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))\).

This data is compatible with the composition and the identity morphisms in the sense that:

- For morphisms \(X \xrightarrow{f} Y \xrightarrow{g} Z\) in \(\mathcal{C}\) we have \(F(g \circ f) = F(g) \circ F(f): F(X) \to F(Z)\) in \(\mathcal{D}\).
- For every object \(X \in \mathcal{C}\) we have \(F(id_X) = id_{F(X)}: F(X) \to F(X)\) in \(\mathcal{D}\).

In order to give some examples of functors, we introduce notation for some prominent categories.

Notation 2.17.

(i) The category of sets and maps of sets will be denoted by \(\textbf{Set}\), the one of pointed sets and maps preserving the chosen elements by \(\textbf{Set}_*\).

(ii) We will write \(\textbf{Grp}\) for the category of groups and group homomorphisms.

(iii) The category of abelian groups and group homomorphisms is denoted by \(\textbf{Ab}\).

(iv) Given a ring \(R\), we write \(\textbf{R-Mod}\) for the category of (left) \(R\)-modules and \(R\)-linear maps.

You know already many examples of functorial constructions. Let us only give a few examples.

Example 2.18.

(i) The formation of free abelian groups generated by a set defines a functor \(\textbf{Set} \to \textbf{Ab}\). More generally, given a ring then there is a free \(R\)-module functor \(\textbf{Set} \to \textbf{R-Mod}\).

(ii) Given a group \(G\) then we obtain the abelianization of \(G\) by dividing out the subgroup generated by the commutators \(aba^{-1}b^{-1}\), \(a, b \in G\). This quotient is an abelian group and gives us a functor \((-)^{ab}: \textbf{Grp} \to \textbf{Ab}: G \mapsto G^{ab}\).
(iii) There are many functors which forget structures or properties. For example there is the following chain of forgetful functors where \( R \) is an arbitrary ring:

\[
R\text{-Mod} \to \text{Ab} \to \text{Grp} \to \text{Set}_* \to \text{Set}
\]

We first forget the action by scalars \( r \in R \) and only keep the abelian group. We then forget the fact that our group is abelian. Next, we drop the group structure and only keep the neutral element. Finally, we also forget the base point.

In this course, many of the constructions on spaces or unpointed spaces turn out to be instances of suitable functors. Let us mention some of the constructions which were already implicit in this course. More examples will be considered in the exercises.

**Example 2.19.**

(i) The fundamental group construction defines a functor \( \pi_1 : \text{Top}_* \to \text{Grp} \).

(ii) The formation of path components defines functors \( \pi_0 : \text{Top} \to \text{Set} \) and \( \pi_0 : \text{Top}_* \to \text{Set}_* \).

(iii) The first two examples are special cases of the following more general construction. Let \( K \) be a space, then we can consider the assignment \( [K, -] : \text{Top} \to \text{Set} : X \mapsto [K, X] \).

Given a map \( f : X \to Y \) of spaces then we obtain an induced map \( [K, X] \to [K, Y] \) by sending a homotopy class \( [g] : K \to X \) to \( [f] \circ [g] : K \to Y \). It is easy to check that this defines a functor \( \text{Top} \to \text{Set} \).

Similarly, if \( (K, k_0) \) is a pointed space, then the assignment \( (X, x_0) \mapsto [(K, k_0), (X, x_0)] \) is functorial. Note that the set \( [(K, k_0), (X, x_0)] \) has a natural base point given by the homotopy class of the constant map \( \kappa_{x_0} : (K, k_0) \to (X, x_0) \). Given a pointed map

\[
f : (X, x_0) \to (Y, y_0)
\]

then the induced map \( [(K, k_0), (X, x_0)] \to [(K, k_0), (Y, y_0)] : [g] \mapsto [f] \circ [g] \) preserves the base point. Thus, we obtain a functor

\[
[(K, k_0), -] : \text{Top}_* \to \text{Set}_*.
\]

The first two examples are obtained by considering the special cases of \( K = * \in \text{Top} \) and \( K = S^0, S^1 \in \text{Top}_* \) respectively.

(iv) The construction of the reduced suspension is functorial and similarly for the loop space. Thus, we have two functors:

\[
\Sigma : \text{Top}_* \to \text{Top}_* \quad \text{and} \quad \Omega : \text{Top}_* \to \text{Top}_*
\]

Let us give some details about the functoriality of \( \Sigma \) (the case of \( \Omega \) will be treated in the exercises). By definition, \( \Sigma(X, x_0) \) is the following quotient space:

\[
X \times S^1 / (\{x_0\} \times S^1 \cup X \times \{\ast\})
\]

If we have a pointed map \( f : (X, x_0) \to (Y, y_0) \), we can form the product with the identity to obtain a map

\[
f \times \text{id}_{S^1} : X \times S^1 \to Y \times S^1
\]

In order to obtain a well-defined map \( \Sigma(f) : \Sigma(X, x_0) \to \Sigma(Y, y_0) \) we want to apply the universal property of the construction of quotient spaces. Thus, it suffices to check that:

\[
(f \times \text{id}_{S^1})(\{x_0\} \times S^1 \cup X \times \{\ast\}) \subseteq \{y_0\} \times S^1 \cup Y \times \{\ast\}
\]
But this is true since \( f \) is a pointed map. Thus, we deduce the existence of a \emph{unique} map \( \Sigma(f) : \Sigma(X, x_0) \to \Sigma(Y, y_0) \) such that the following square commutes:

\[
\begin{array}{ccc}
X \times S^1 & \longrightarrow & Y \times S^1 \\
\downarrow & & \downarrow \\
\Sigma(X, x_0) & \longrightarrow & \Sigma(Y, y_0)
\end{array}
\]

We leave it as an exercise to check that the uniqueness implies that we indeed get a functor \( \Sigma : \text{Top}_* \to \text{Top}_* \). For example, you might want to consider the following diagram to prove the compatibility with respect to compositions:

\[
\begin{array}{ccc}
X \times S^1 & \longrightarrow & Y \times S^1 & \longrightarrow & Z \times S^1 \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma(X, x_0) & \longrightarrow & \Sigma(Y, y_0) & \longrightarrow & \Sigma(Z, z_0)
\end{array}
\]

(v) By adding disjoint base points to spaces we obtain a functor \( (\cdot)_+ : \text{Top} \to \text{Top}_* \). There is also a forgetful functor \( \text{Top}_* \to \text{Top} \) in the opposite direction which forgets the base points.
LECTURE 3: HIGHER HOMOTOPY GROUPS

In this section we will introduce the main objects of study of this course, the homotopy groups

\[ \pi_n(X,x_0) \]

of a pointed space \((X,x_0)\), for each natural number \(n \geq 2\). (Recall that the (pointed) set of components \(\pi_0(X,x_0)\) and the fundamental group \(\pi_1(X,x_0)\) have already been defined.) One goal of this course is to develop some techniques which will allow us to calculate these homotopy groups in interesting examples.

1. HIGHER HOMOTOPY GROUPS

We begin by introducing some notation for important spaces. Let us denote by

\[ I^n = [0,1] \times \ldots \times [0,1] \subseteq \mathbb{R}^n \]

the \(n\)-cube and \(\partial I^n \subseteq I^n\) for its boundary. Thus,

\[ \partial I^n = \{(t_1, \ldots, t_n) \in I^n \mid \text{at least one of the } t_i \in \{0,1\} \} \]

Let us agree on the convention that \(\partial I^0 = \emptyset\) is empty. Note that the boundary satisfies (and is completely determined by \(\partial I^n\) and) the Leibniz formula

\[ \partial(I^n \times I^m) = (\partial I^n) \times I^m \cup I^n \times (\partial I^m) \]

The \(n\)-sphere is denoted by

\[ S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \]

Note that there are homeomorphisms \(I^n / \partial I^n \cong S^n\); we will write \([t_1, \ldots, t_n] \in S^n\) for the image of \((t_1, \ldots, t_n) \in I^n\) under the composition \(I^n \to I^n / \partial I^n \cong S^n\).

The definition of the underlying (pointed) set of \(\pi_n(X,x_0)\) is simple enough:

\[ \pi_n(X,x_0) = \left\{\left[(I^n, \partial I^n), (X,x_0)\right]\right\} \]

Thus, an element \([\alpha]\) of \(\pi_n(X,x_0)\) is represented by a map \(\alpha: I^n \to X\) sending the entire boundary \(\partial I^n\) to the base point \(x_0\); and two such \(\alpha\) and \(\alpha'\) represent the same element of \(\pi_n(X,x_0)\) if and only if there is a homotopy \(H: I^n \times I \to X\) such that

\[ H(\partial I^n \times I) = x_0, \quad H(-,0) = \alpha, \quad \text{and} \quad H(-,1) = \alpha'. \]

Obviously, a map \(f: (X,x_0) \to (Y,y_0)\) induces a function

\[ f_*: \pi_n(X,x_0) \to \pi_n(Y,y_0) \]

which in fact only depends on the homotopy class of \(f\). Just like for the fundamental group, we have

**Proposition 3.1.**

(i) For each pointed space \((X,x_0)\) and \(n \geq 1\), the set \(\pi_n(X,x_0)\) is a group, the \(n\)-th homotopy group of \((X,x_0)\).
Lecture 3: Higher Homotopy Groups

(i) For each map \((X,x_0) \to (Y,y_0)\), the induced operation \(\pi_n(X,x_0) \to \pi_n(Y,y_0)\) is a group homomorphism, defining a functor \(\pi_n: \text{Top} \to \text{Grp}\). The functor \(\pi_n\) is homotopy invariant, i.e., \(\pi_n(f) = \pi_n(g)\) for homotopic maps \(f \simeq g\).

**Proof.** For two elements \([\alpha]\) and \([\beta]\) in \(\pi_n(X,x_0)\), the product \([\beta] \circ [\alpha]\) is represented by the map \(\beta \ast \alpha: I^n \to X\) defined by:

\[
(\beta \ast \alpha)(t_1, \ldots, t_n) = \begin{cases} 
\alpha(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\
\beta(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \leq t_1 \leq 1
\end{cases}
\]

Notice that the definition agrees with the known group structure on the fundamental group for \(n = 1\). The proof that \(\circ\) is well-defined and associative, that the constant map \(\kappa_{x_0}: I^n \to X\) represents a neutral element, and that each element \([\alpha]\) has an inverse represented by

\[
\alpha^{-1}(t_1, \ldots, t_n) = \alpha(1 - t_1, t_2, \ldots, t_n)
\]

is exactly the same as for \(\pi_1\), and we leave the details as an exercise. Also the functoriality is an exercise. \(\square\)

**Remark 3.2.** Let \(X\) be a space and let \(x_0, x_1 \in X\). In general, \(\pi_n(X,x_0)\) and \(\pi_n(X,x_1)\) can be very different. In fact, homotopy groups only `see the path-component of the base point'. More precisely, let \((X,x_0)\) be a pointed space and let \(X' = [x_0]\) be the path-component of \(x_0\). Then the inclusion \(i: (X',x_0) \to (X,x_0)\) induces an isomorphism \(i_*: \pi_n(X',x_0) \to \pi_n(X,x_0)\) for all \(n \geq 1\). This follows immediately from the fact that \(S^n\) is path-connected for \(n \geq 1\). We will see later that any path between two points \(x_0, x_1 \in X\) induces an isomorphism \(\pi_n(X,x_0) \cong \pi_n(X,x_1)\).

**Remark 3.3.**

(i) Of course it is not only the validity of the proposition which is important, but also the explicit description of the product. However, it can be shown that this group structure is unique for \(n \geq 2\).

(ii) We said that the proof of the group structure is analogous to the argument for \(\pi_1\). In fact, there is a more formal way to see this, as we will see below.

One may object that the definition of the group structure is a bit unnatural, because the first coordinate \(t_1\) is given a preferred rôle in the definition of the group structure. We could also define a product as follows:

\[
(\beta \ast_i \alpha)(t_1, \ldots, t_n) = \begin{cases} 
\alpha(t_1, \ldots, 2t_1, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\
\beta(t_1, \ldots, 2t_1 - 1, \ldots, t_n) & 1/2 \leq t_1 \leq 1
\end{cases}
\]

The explanation is that these two products induce the same operation on homotopy classes. The proof of this fact is given by the following observation (Lemma 3.4) together with the so-called Eckmann-Hilton argument (Proposition 3.3).

**Lemma 3.4.** The operation \(\ast\) distributes over the operation \(\ast_i\) in the sense that

\[
(\alpha \ast_i \beta) \ast (\gamma \ast_i \delta) = (\alpha \ast \gamma) \ast_i (\beta \ast \delta)
\]

for all maps \(\alpha, \beta, \gamma, \delta: (I^n, \partial I^n) \to (X,x_0)\).

**Proof.** We only have to look at the case \(n = 2, i = 2\). Then the expressions on the left and right correspond to the same subdivisions of the square so define identical maps (draw the picture!). \(\square\)
Proposition 3.5. (‘Eckmann-Hilton trick’)
Let $S$ be a set with two operations $\bullet, \circ : S \times S \to S$ having a common unit $e \in S$. Suppose $\bullet$ and $\circ$ distribute over each other, in the sense that
$$(\alpha \bullet \beta) \circ (\gamma \bullet \delta) = (\alpha \circ \gamma) \bullet (\beta \circ \delta)$$
Then $\bullet$ and $\circ$ coincide, and define a commutative and associative operation on $S$.

Proof. Taking $\beta = e = \gamma$ in the distributive law yields $\alpha \circ \delta = \alpha \bullet \delta$, while taking $\alpha = e = \delta$ yields $\beta \circ \gamma = \gamma \bullet \beta$. The associativity is obtained by taking $\beta = e$ in the distributive law. \hfill $\square$

Applying this proposition to $\ast$ and $*_i$ shows that these define the same operation on $\pi_n(X,x_0)$ for $n \geq 2$. The proposition also shows:

Corollary 3.6. The groups $\pi_n(X,x_0)$ are abelian for $n \geq 2$.

Remark 3.7. For this reason, one often employs additive notation for the group structure on $\pi_n(X,x_0)$, writing:
$$[\beta] + [\alpha] = [\beta \ast \alpha]$$
$$0 = [k_{x_0}]$$
$$-[\alpha] = [\alpha^{-1}]$$

There is yet another way of describing $\pi_n(X,x_0)$.

Proposition 3.8.
(i) There is a bijection of sets natural in the pointed space $(X,x_0)$:
$$[(S^n,\ast),(X,x_0)] \cong \pi_n(X,x_0)$$
(ii) The group structure on $[(S^n,\ast),(X,x_0)]$ induced by this bijection coincides with the one obtained by composition with the ‘pinch map’
$$\nabla : S^n \to S^n \vee S^n$$
defined by collapsing the equator in $S^n$ to a single point.

Proof. Part (i) follows immediately from the isomorphism
$$(I^n/\partial I^n,\ast) \to (S^n,\ast).$$

For part (ii), recall from the exercises that the wedge $\vee$ defines a coproduct in the category of pointed spaces, so that two maps $\alpha,\beta : (S^n,\ast) \to (X,x_0)$ together uniquely define a map
$$\alpha \vee \beta : (S^n \vee S^n,\ast) \to (X,x_0).$$

Thus we get an induced operation on $[(S^n,\ast),(X,x_0)]$ defined by
$$\beta \ast \alpha = (\alpha \vee \beta) \circ \nabla$$

It is easy to check that this corresponds to the operation $\ast$ on maps from $(I^n,\partial I^n)$, once one takes the equator in $S^n$ to be the image of $\{t_1 = 1/2\} \subseteq I^n$ under the map $I^n \to I^n/\partial I^n \cong S^n$. \hfill $\square$

Example 3.9. For the one-point space $\ast \in \text{Top}_*$ there is precisely one pointed map $S^n \to \ast$ for each $n \geq 0$. Thus we have
$$\pi_0(\ast) \cong \ast, \quad \pi_1(\ast) \cong 1, \quad \text{and} \quad \pi_n(\ast) \cong 0, \quad n \geq 2.$$ We will refer to this by saying that $\pi_n(\ast)$ is trivial for all $n \geq 0$.

This example together with the homotopy invariance immediately gives us the following.
Corollary 3.10. Let \( f: (X, x_0) \to (Y, y_0) \) be a pointed homotopy-equivalence. Then the induced map
\[
f_*: \pi_n(X, x_0) \to \pi_n(Y, y_0)
\]
is an isomorphism. In particular, \( \pi_n(X, x_0) \) is trivial for all choices of base points in a contractible space \( X \) and all \( n \geq 0 \).

Proof. Let \( g: (Y, y_0) \to (X, x_0) \) be an inverse pointed homotopy equivalence so that we have:
\[
g \circ f \simeq \text{id}_X \quad \text{rel } x_0 \quad \text{and} \quad f \circ g \simeq \text{id}_Y \quad \text{rel } y_0
\]
Homotopy invariance gives \( g_* \circ f_* = (g \circ f)_* = \text{id} \), and similarly \( f_* \circ g_* = \text{id} \). For the second claim, given a contractible space \( X \) and \( x_0 \in X \), it suffices to consider the pointed homotopy equivalence \((X, x_0) \to ∗\). \( \square \)

We will later see that these groups \( \pi_n(X, x_0) \) are non-trivial and highly informative, but we need to develop (or know) a little more theory before we can make this precise. However, assuming a bit of background knowledge, we observe the following.

Example 3.11. (Preview of examples)
(i) The identity map \( \text{id}: S^n \to S^n \) defines an element of \( \pi_n(X, x_0) \). If you know something about degrees, you know that the constant map has degree zero while the identity has degree 1. It is a fact that the degrees of homotopic maps coincide, we conclude that \( 0 \neq [\text{id}] \) in \( \pi_n(S^n, ∗) \). In fact, we will show that there is an isomorphism \( \pi_n(S^n, ∗) \cong \mathbb{Z} \) and that \( [\text{id}] \) is a generator. This could be proved, e.g., using singular homology but we will obtain this calculation as a consequence of the homotopy excision theorem.

(ii) If you know a bit of differential topology then you know that any map \( S^k \to S^n \) is homotopic to a smooth map, and that a smooth map \( f: S^k \to S^n \) cannot be surjective if \( k < n \). So such a map \( f \) factors as a composition
\[
S^k \to S^n - \{x\} \cong \mathbb{R}^n \to S^n
\]
for some point \( x \in S^n \) not in the image of \( f \). The contractibility of \( \mathbb{R}^n \) implies that \( f \) is homotopic to a constant map. Thus,
\[
\pi_k(S^n, ∗) \cong 0, \quad k < n.
\]
We will later deduce this result from the cellular approximation theorem.

(iii) Consider the scalar multiplication on the complex vector space of dimension 2,
\[
\mu: \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2.
\]
When we restrict to the complex numbers, respectively vectors of norm 1, we obtain a map
\[
\mu: S^1 \times S^3 \to S^3,
\]
an action of the circle-group on the 3-sphere. It can be shown that the orbit space of this action is \( S^2 \). The quotient map \( S^3 \to S^2 \) is the famous Hopf fibration, and defines a non-zero element in \( \pi_3(S^2, ∗) \).
2. H-spaces and H-groups

Let us now examine loop spaces in some more detail. Recall the construction of the loop space \( \Omega(X, x_0) \) associated to a pointed space \((X, x_0)\), and the isomorphism

\[
\pi_1(X, x_0) \cong \pi_0(\Omega(X, x_0)).
\]

The group structure on \( \pi_1(X, x_0) \) comes from a ‘group structure up to homotopy’ on \( \Omega(X, x_0) \). Explicitly, writing \( H = \Omega(X, x_0) \) and \( e = \kappa_{x_0} \in H \) for the constant loop at \( x_0 \), there is a multiplication map on \( H \) given by the concatenation:

\[
\mu : H \times H \to H : (\beta, \alpha) \mapsto \beta \ast \alpha
\]

This multiplication is associative up to homotopy in the sense that the following two maps are homotopic:

\[
\begin{array}{ccc}
H \times H \times H & \xrightarrow{id \times \mu} & H \times H \\
\downarrow & & \downarrow \mu \\
H & \xrightarrow{\mu} & H
\end{array}
\]

\[
\begin{array}{ccc}
H \times H \times H & \xrightarrow{id \times \mu} & H \times H \\
\downarrow & & \downarrow \mu \\
H & \xrightarrow{\mu} & H
\end{array}
\]

Moreover, this multiplication is unital up to homotopy, i.e., we have homotopies from \( id \) to \( \mu \circ (e \times id) \) and \( \mu \circ (id \times e) \):

\[
\begin{array}{ccc}
H & \xrightarrow{\mu} & H \\
\downarrow & & \downarrow \mu \\
H & \xrightarrow{e} & H \\
\downarrow & & \downarrow \mu \\
H & \xrightarrow{id} & H
\end{array}
\]

Indeed, these homotopies are given by the usual reparametrization homotopies. A pointed space \((H, e)\) with such an additional structure is called an (associative) H-space (or Hopf space).

Given any such H-space \((H, e)\), composition with \( \mu \) defines an associative multiplication on the set of homotopy classes of maps \([[(Y, y_0), (H, e)]\]) for an arbitrary pointed space \((Y, y_0)\). Moreover, this ‘multiplicative structure’ is natural in \((Y, y_0)\). (If you do not know what we mean by this naturality, then see the exercise sheet.)

Moreover, the associative multiplication defines a group structure on this set if \( H \) has a homotopy inverse, i.e., if there is a map \( i : H \to H \) such that the following diagram commutes up to homotopies:

\[
\begin{array}{ccc}
H & \xrightarrow{\mu} & H \\
\downarrow & & \downarrow \mu \\
H & \xrightarrow{i \times id} & H \\
\downarrow & & \downarrow \mu \\
H & \xrightarrow{id \times i} & H
\end{array}
\]

\[
\begin{array}{ccc}
\alpha & \xrightarrow{(i(\alpha), \alpha)} & (\alpha, i(\alpha)) \\
\downarrow & & \downarrow \\
\alpha * \alpha & \xleftarrow{i(\alpha) \ast \alpha} & \alpha * i(\alpha)
\end{array}
\]

In this case \((H, e)\) is called an H-group. Thus, \( \Omega(X, x_0) \) is an H-group, and for each pointed space \((Y, y_0)\) the set \([[(Y, y_0), \Omega(X, x_0)]\}) carries a natural group structure. For the one-point space \(* \in \text{Top}_*\),
this defines the usual group structure on:
\[ \ast, \Omega(X, x_0) \cong \pi_0(\Omega(X, x_0)) \cong \pi_1(X, x_0) \]

**Exercise 3.12.** Define the notion of a commutative H-space and a commutative H-group. Is the loop space of a pointed space with the concatenation pairing a commutative H-group?

**Theorem 3.13.** For every \( n \geq 1 \), there is a natural isomorphism of groups:
\[ \pi_n(X, x_0) \cong \pi_{n-1}(\Omega(X, x_0)) \]

Let us begin with a lemma. Recall our notation \([x, y] \in X \wedge Y\) for points in a smash product. Moreover, let us use a similar notation for elements in a quotient space, i.e., we will write \([x] \in X/A\).

For convenience, we will drop base points from notation in the next lemma and we use \(I^n/\partial I^n\) as our model for the \(n\)-sphere.

**Lemma 3.14.** The following map is a pointed homeomorphism:
\[ S^n \wedge S^m \rightarrow S^{n+m} : \left[[t_1, \ldots, t_n], [t'_1, \ldots, t'_m]\right] \mapsto [t_1, \ldots, t_n, t'_1, \ldots, t'_m] \]

**Proof.** One checks directly that this is a well-defined continuous bijection between compact Hausdorff spaces and hence a homeomorphism. \(\square\)

This map can be described as:
\[
S^n \wedge S^m = (I^n/\partial I^n) \wedge (I^m/\partial I^m)
\]
\[
= (I^n/\partial I^n \times I^m/\partial I^m)/(\ast \times I^m/\partial I^m \cup I^n/\partial I^n \times \ast)
\]
\[
\cong I^n \times I^m/(\partial I^n \times I^m \cup I^n \times \partial I^m)
\]
\[
\cong I^{n+m}/\partial I^{n+m}
\]
\[
= S^{n+m}
\]

**Proof.** (of Theorem 3.13). The multiplication on \(\pi_{n-1}(\Omega(X, x_0))\) is the ‘loop multiplication’. We already know from an earlier lecture that there is a natural isomorphism:
\[
[(S^{n-1}, \ast), \Omega(X, x_0)] \cong [(S^{n-1} \wedge S^1, \ast), (X, x_0)]
\]

Using the homeomorphism of the above lemma, it is immediate that the pairing on \([(S^n, \ast), (X, x_0)]\) induced by the loop multiplication is \(\ast_n\), the ‘concatenation with respect to the last coordinate’. But we already know that this is identical to the group structure on \(\pi_n(X, x_0)\). \(\square\)

If we look at the theorem for \(n \geq 2\), we see that \(\pi_{n-1}(\Omega(X, x_0))\) has two group structures: one is the group structure on \(\pi_{n-1}\) for any pointed space, and the other is an instance of the group structure on \([(Y, y_0), (H, e)]\) for any H-group \(H\), in this case for \(Y = S^{n-1}\) and \(H = \Omega(X, x_0)\). Moreover, these group structures distribute over each other. Indeed, the multiplication \(\mu: H \times H \rightarrow H\) induces a group homomorphism
\[
\mu_\ast: \pi_{n-1}(H, e) \times \pi_{n-1}(H, e) \rightarrow \pi_{n-1}(H, e)
\]
and this precisely means that the multiplication coming from the H-group distributes over the one coming from \(\pi_{n-1}\). So, by Eckmann-Hilton (Proposition 3.5), the two multiplications coincide and are commutative.

**Corollary 3.15.** The fundamental group \(\pi_1(H, e)\) of an H-group \((H, e)\) is abelian.
We now come to a different model for the loop space. We have seen that $\Omega(X,x_0)$ has a multiplication which is associative and unital 'up to homotopy'. One may wonder whether there is a way to make this multiplication strictly associative and unital. For a general H-space this need not be possible. But in this special case there is an easy way to do this. Let $M(X,x_0)$ be the space of Moore loops (named after J. C. Moore). Its points are pairs $(t,\alpha)$ where $t \in \mathbb{R}$, $t \geq 0$, and $\alpha: [0,t] \to X$ is a loop at $x_0$ of length $t$ (i.e., $\alpha(0) = x_0 = \alpha(t)$). We can topologize this set as a subspace of $\mathbb{R} \times X^{[0,\infty)}$, identifying a path $\alpha: [0,t] \to X$ with the map $[0,\infty) \to X$ which is constant on $[t,\infty)$, and the resulting space is the Moore loop space. Then there is a continuous and strictly associative multiplication on $M(X,x_0)$, given by

$$(t, \beta) \cdot (s, \alpha) = (t + s, \beta \ast_M \alpha)$$

where:

$$(\beta \ast_M \alpha)(r) = \begin{cases} \alpha(r) & 0 \leq r \leq s \\ \beta(r - s) & s \leq r \leq t + s \end{cases}$$

A strict unit for this multiplication is $(0, \kappa_{x_0})$.

The space $M(X,x_0)$ is homotopy equivalent to $\Omega(X,x_0)$. Indeed there are maps

$$\Omega(X,x_0) \xrightarrow{\psi} M(X,x_0),$$

$\psi$ is simply the inclusion, while $\phi$ is defined by

$$\phi(t, \alpha)(r) = \alpha(t \cdot r), \quad 0 \leq r \leq 1.$$ 

Then obviously $\phi \circ \psi$ is the identity, while

$$H_s(t, \alpha) = \left( \frac{t}{(1 - s) + st}, \alpha((1 - s) + st) \cdot - \right)$$

defines a homotopy from $H_1 = \psi \circ \phi$ to the identity $H_0$.

**Remark 3.16.** We just observed that the loop space and the Moore loop space are homotopy equivalent spaces. Note that the respective H-group structures correspond to each other under these homotopy equivalences. However the multiplications have different formal properties: the Moore loop space is strictly associative while the loop space is only associative up to homotopy. Thus we see that a space $X$ homotopy equivalent to a space with a strictly associative multiplication does not necessarily inherit the same structure. But it is easy to see that $X$ can be turned into an H-space that way. To put it as a slogan:

‘strictly associative multiplications do not live in homotopy theory’

As we already mentioned not all H-spaces can be rectified in the sense that they would be homotopy equivalent to spaces with a strictly associative multiplication. One might wonder what additional structure would be needed for this to become true. There is an answer to this question lying beyond the scope of these lectures. Nevertheless, these questions and the more general search for homotopy invariant algebraic structures initiated the development of a good deal of mathematics.

Let us formalize the notion of a homotopy invariant functor. Let $\mathcal{C}$ be an arbitrary category. Then a functor $F: \mathbf{Top}_* \to \mathcal{C}$ is homotopy invariant if pointed maps which are homotopic relative to the base point always have the same image under $F$:

$$f \simeq g \quad \text{implies} \quad F(f) = F(g)$$
Now, note that there is a canonical functor
\[ \gamma : \text{Top}_* \to \text{Ho}(\text{Top}_*) \]
which is the identity on objects and which sends a pointed map to its pointed homotopy class.

**Exercise 3.17.**

(i) The above assignments, in fact, define a functor \( \gamma : \text{Top}_* \to \text{Ho}(\text{Top}_*) \) and this functor is homotopy invariant.

(ii) Let \( \mathcal{C} \) be a category. A functor \( F : \text{Top}_* \to \mathcal{C} \) is homotopy invariant if and only if there is a functor \( F' : \text{Ho}(\text{Top}_*) \to \mathcal{C} \) such that \( F = F' \circ \gamma : \text{Top}_* \to \mathcal{C} \). In this case the functor \( F' \) is unique.

(iii) Redo a similar reasoning for the categories \( \text{Top} \) and \( \text{Top}^2 \).

Thus a homotopy invariant functor ‘is the same thing’ as a functor defined on the homotopy category of (pointed or pairs of) spaces. In particular, we have:

\[ \pi_0 : \text{Ho}(\text{Top}_*) \to \text{Set}_* , \quad \pi_1 : \text{Ho}(\text{Top}_*) \to \text{Grp} , \quad \text{and} \quad \pi_n : \text{Ho}(\text{Top}_*) \to \text{Ab} \]
LECTURE 4: RELATIVE HOMOTOPY GROUPS
AND THE ACTION OF THE FUNDAMENTAL GROUP

In this section we will introduce relative homotopy groups of a (pointed) pair of spaces. Associated to such a pair we obtain a long exact sequence in homotopy relating the absolute and the relative groups. This and related long exact sequences are useful in calculations as we will see later. Moreover, we want to clarify the role played by the choice of base points. Expressed in a fancy way, we will show that the assignment \( x_0 \mapsto \pi_n(X, x_0) \) defines a functor on the fundamental groupoid \( \pi(X) \) of \( X \). This encodes, in particular, an action of the fundamental group on higher homotopy groups.

1. Relative homotopy groups

To begin with let us consider a pointed space \((X, x_0)\) and a subspace \(A \subseteq X\) containing the base point \(x_0\). Thus we have an inclusion of pointed spaces \(i: (A, x_0) \to (X, x_0)\) and we refer to \((X, A, x_0)\) as a pointed pair of spaces. The inclusion induces a map at the level of homotopy groups (or sets)

\[ i_\ast: \pi_n(A, x_0) \to \pi_n(X, x_0), \quad n \geq 0. \]

which, in general, is not injective. A homotopy class \( \alpha \in \pi_n(A, x_0) \) lies in the kernel of \( i_\ast \) if for any map \( f: (I^n, \partial I^n) \to (A, x_0) \) representing it the induced map \( i \circ f: (I^n, \partial I^n) \to (X, x_0) \) is homotopic to the constant map \( \kappa_{x_0} \). Such a homotopy is a map \( H: I^n \times I \to X \) satisfying the following relations:

\[ H(-, 1) = f, \quad H(-, 0) = \kappa_{x_0}, \quad \text{and} \quad H|_{\partial I^n \times I} = \kappa_{x_0}. \]

Thus, if we denote by \( J^n \) the subspace of the boundary \( \partial I^{n+1} = \partial I^n \cup \partial I^n \times I \) given by

\[ J^n = I^n \times \{0\} \cup \partial I^n \times I \]

then such a homotopy is a map of triples of spaces (in the obvious sense):

\[ H: (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0) \]

There is also an adapted notion of homotopies of maps of triples which we want to introduce in full generality. Let \( X_2 \subseteq X_1 \subseteq X_0 \) and \( Y_2 \subseteq Y_1 \subseteq Y_0 \) be triples of spaces and let

\[ f, g: (X_0, X_1, X_2) \to (Y_0, Y_1, Y_2) \]

be maps of triples. Then a homotopy \( H: f \simeq g \) is a map of triples

\[ H: (X_0, X_1, X_2) \times I = (X_0 \times I, X_1 \times I, X_2 \times I) \to (Y_0, Y_1, Y_2) \]

which satisfies \( H(-, 0) = f \) and \( H(-, 1) = g \). Thus, we are asking for a homotopy \( H: X_0 \times I \to Y_0 \) which has the property that each map \( H(-, t) \) respects the subspace inclusions, i.e., is a map of triples \( H(-, t): (X_0, X_1, X_2) \to (Y_0, Y_1, Y_2) \). In the special case that \( X_2 \) and \( Y_2 \) are just base points, this gives us the notion of homotopies of maps of pointed pairs.

**Exercise 4.1.** This homotopy relation is an equivalence relation which is well-behaved with respect to maps of triples. Similarly, we get such a result for pointed pairs of spaces. There are homotopy categories of triples of spaces and pointed pairs of spaces.
Maybe you should not carry out this exercise in detail but only play a bit with the notions in order to convince yourself that they behave as expected.

Now, back to our pointed pair \((X, A, x_0)\). The above discussion motivates the following definition:

\[
\pi_n(X, A, x_0) = \left[\left(\partial I^n, \partial I^n, J^{n-1}\right), (X, A, x_0)\right], \quad n \geq 1
\]

(Note that in the case of \(A = \{x_0\}\) we have \(\pi_n(X, x_0, x_0) = \pi_n(X, x_0)\).) A priori, the \(\pi_n(X, A, x_0)\) are only pointed sets, the base point being given by the homotopy class of the constant map \(\kappa_{x_0}\).

However, it turns out that we get groups for \(n \geq 2\) which are abelian for \(n \geq 3\). To this end let us consider maps

\[
\alpha, \beta: \left(\partial I^n, \partial I^n, J^{n-1}\right) \to (X, A, x_0), \quad n \geq 2
\]

Then we can define the concatenation \(\beta \ast \alpha: \left(\partial I^n, \partial I^n, J^{n-1}\right) \to (X, A, x_0)\) by the ‘usual formula’:

\[
(\beta \ast \alpha)(t_1, \ldots, t_n) = \begin{cases} 
\alpha(2t_1, t_2, \ldots, t_n), & 0 \leq t_1 \leq 1/2 \\
\beta(2t_1 - 1, t_2, \ldots, t_n), & 1/2 \leq t_1 \leq 1
\end{cases}
\]

It follows immediately that \(\beta \ast \alpha\) again is a map of triples. As in earlier lectures one checks that this concatenation is well-defined on homotopy classes and defines a group structure on \(\pi_n(X, A, x_0)\) with neutral element given by the homotopy class of the constant map.

**Definition 4.2.** Let \((X, A, x_0)\) be a pointed pair of spaces. Then the group

\[
\pi_n(X, A, x_0) = \left[\left(\partial I^n, \partial I^n, J^{n-1}\right), (X, A, x_0)\right], \quad n \geq 2,
\]

is the \(n\)-th relative homotopy group of \((X, A, x_0)\). The pointed set

\[
\pi_1(X, A, x_0) = \left[\left(\partial I^1, \partial I^1, 0\right), (X, A, x_0)\right]
\]

is the first relative homotopy set of \((X, A, x_0)\).

To avoid awkward notation we will simply write \(\pi_n(X, A)\) instead of \(\pi_n(X, A, x_0)\) unless there is a risk of ambiguity. Now, if \(n \geq 3\) one could again object that the above definition for the concatenation is not very natural. In fact, one could also define pairings \(*_i\), where \(1 \leq i \leq n - 1\), given by the formula:

\[
(\beta *_i \alpha)(t_1, \ldots, t_n) = \begin{cases} 
\alpha(t_1, \ldots, 2t_i, \ldots, t_n), & 0 \leq t_i \leq 1/2 \\
\beta(t_1, \ldots, 2t_i - 1, \ldots, t_n), & 1/2 \leq t_i \leq 1
\end{cases}
\]

(Note that there is no \(*_n\) unless \(A = \{x_0\}\) and this is why \(\pi_1(X, A)\) is only a pointed set in general.) Following the lines of the last lecture (‘Eckmann-Hilton trick’) one checks that these different pairings induce the same group structure and that \(\pi_n(X, A)\) is abelian for \(n \geq 3\). If we denote by \(\text{Top}_2^*\) the category of pointed pairs of spaces, then our discussion gives us the following:

**Corollary 4.3.** The assignments \((X, A, x_0) \mapsto \pi_n(X, A)\) can be extended to define functors:

\[
\pi_1: \text{Top}_2^* \to \text{Set}_*, \quad \pi_2: \text{Top}_2^* \to \text{Grp}, \quad \text{and} \quad \pi_n: \text{Top}_2^* \to \text{Ab}, \quad n \geq 3
\]

**Exercise 4.4.** Convince yourself that \((X, A, x_0) \mapsto \pi_2(X, A)\) really defines a functor taking values in groups by drawing some diagrams. If you are ambitious, then do similarly in order to see that \(\pi_n(X, A)\) always is an abelian group.

A different way of proving this corollary is sketched in the exercises. There, you will show that \(\pi_{n+1}(X, A)\) is naturally isomorphic to the \(n\)-th homotopy group of a certain space \(P(X; x_0, A)\).
2. The long exact sequence for relative homotopy groups

The motivation for this discussion was the observation that an inclusion \( i: (A, x_0) \to (X, x_0) \) induces a morphism of homotopy groups which is not necessarily injective. The relative homotopy groups are designed to measure the deviation from this. In fact, if \( j \) denotes the inclusion \( j: (X, x_0) \to (X, A) \) then there is the following result.

**Proposition 4.5.** Given a pointed pair of spaces \((X, A, x_0)\), there are connecting homomorphisms \( \partial: \pi_n(X, A) \to \pi_{n-1}(A, x_0) \), \( n \geq 1 \), such that the following sequence is exact:

\[
\ldots \to \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \ldots \xrightarrow{i_*} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)
\]

This is the long exact homotopy sequence of the pointed pair \((X, A, x_0)\). Moreover, this sequence is natural in the pointed pair.

Before we attack the proof let us be a bit more precise about the statement. Recall that a diagram of groups and group homomorphisms \( G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \) is exact at \( G_2 \) if we have the equality \( \text{im}(f) = \ker(g) \) of subgroups of \( G_2 \). In particular, the composition \( g \circ f \) sends everything to the neutral element of \( G_3 \), but we also have a converse inclusion. Namely, if \( x_2 \in G_2 \) lies in \( \ker(g) \), then it already comes from \( G_1 \), i.e., there is an element \( x_1 \in G_1 \) such that \( f(x_1) = x_2 \).

More generally, a diagram of groups and group homomorphisms

\[ G_1 \to G_2 \to \ldots \to G_{n-1} \to G_n \]

is exact if it is exact at \( G_i \) for all \( 2 \leq i \leq n - 1 \). A special case is a short exact sequence which is an exact diagram of the form:

\[ 1 \to G_1 \to G_2 \to G_3 \to 1 \]

**Example 4.6.** Let \( G \) and \( H \) be groups.

(i) A homomorphism \( G \to H \) is injective if and only if \( 1 \to G \to H \) is exact.

(ii) A homomorphism \( G \to H \) is surjective if and only if \( G \to H \to 1 \) is exact.

(iii) A homomorphism \( G \to H \) is an isomorphism if and only if \( 1 \to G \to H \to 1 \) is exact.

(iv) A group \( G \) is trivial if and only if \( 1 \to G \to 1 \) is exact.

In particular, a short exact sequence basically encodes a surjective homomorphism \( G_2 \to G_3 \) together with the inclusion of the kernel \( N = G_1 \subseteq G_2 \).

Now, in the diagram we consider in the above proposition not all maps are homomorphisms of groups. In fact, the last three entries \( \pi_1(X, A) \), \( \pi_0(A, x_0) \), and \( \pi_0(X, x_0) \) are only pointed sets. The notion of exactness is extended to the context of maps of pointed sets by defining the kernel of such a map to be the preimage of the base point.

Finally, let us make precise the meaning of the naturality in the above proposition. If we have a map of pointed pairs \( f: (X, A, x_0) \to (Y, B, y_0) \), then we have a connecting homomorphism for each of the pointed pairs. The naturality means that the following square commutes:

\[
\begin{array}{ccc}
\pi_{n+1}(X, A) & \xrightarrow{\partial} & \pi_n(A, x_0) \\
\downarrow f_* & & \downarrow f_* \\
\pi_{n+1}(Y, B) & \xrightarrow{\partial} & \pi_n(B, y_0)
\end{array}
\]
It is easy to check that from this we actually get a commutative ladder of the form:

\[ \ldots \rightarrow \pi_1(X, x_0) \rightarrow \pi_1(X, A) \rightarrow \pi_0(A, x_0) \rightarrow \pi_0(X, x_0) \]

For the purpose of the following lemma let us introduce some notation. Recall that \(J^n\) is obtained from \(I^{n+1}\) by removing the ‘interior of the cube and the interior of the top face’. From a different perspective \(J^n\) is obtained from \(I^n = I^n \times \{0\}\) by gluing a further copy of \(I^n\) on each face of \(\partial I^n\).

Now, if \(F\) is a collection of faces of \(I^n\), then let \(J^n_F \subseteq J^n\) be obtained from \(I^n\) by gluing only those copies of \(I^n\) which correspond to faces in \(F\). More formally, the cube \(I^n \subseteq \mathbb{R}^n\) has \(2n\) faces. These can be restricted to the top face \(\partial I^n\) and to check that this gives us the intended relative homotopy. Thus the face \(J^n_F \subseteq I^n\) corresponding to an index \(f = (j, i_j)\) is given by:

\[ J^n_F = \{ (t_1, \ldots, t_n) \in I^n \mid t_j = i_j \} \]

With this notation the space \(J^n_F \subseteq J^n \subseteq \partial I^{n+1}\) associated to a set \(F\) of faces is given by:

\[ J^n_F = I^n \times \{0\} \cup \bigcup_{f \in F} I^{-1}_f \times I \]

**Lemma 4.7.**

(i) The map \(i: J^{n-1} \rightarrow I^n\) is the inclusion of a strong deformation retract, i.e., there is a map \(r: I^n \rightarrow J^{n-1}\) which satisfies \(r \circ i = id_{J^{n-1}}\) and \(i \circ r \simeq id_{I^n}\) (rel \(J^{n-1}\)).

(ii) Given a set \(F\) of faces of \(I^{n-1}\) then \(J^n_F \subseteq I^n\) is the inclusion of a strong deformation retract.

**Proof.** We will only give the proof of the first claim, the second one is an exercise. If we consider the space \(I^n \subseteq \mathbb{R}^n\) as the unit cube of length one, then let \(s\) be the point \(s = (1/2, \ldots, 1/2, 2)\) sitting ‘above the center of the cube’. For each point \(x \in I^n\) let \(l(x)\) be the unique line in \(\mathbb{R}^n\) passing through \(s\) and \(x\). This line \(l(x)\) intersects \(J^{n-1}\) in a unique point which we take as the definition of \(r(x)\). It is easy to see that the resulting map \(r: I^n \rightarrow J^{n-1}\) is continuous and that we have \(r \circ i = id\). The homotopy \(i \circ r \simeq id\) (rel \(J^{n-1}\)) is obtained by ‘collapsing the line segments between \(x\) and \(r(x)\)’ to \(r(x)\). We leave it to the reader to write down an explicit formula for this and to check that this gives us the intended relative homotopy.

With this preparation we can now turn to the proof of the proposition.

**Proof.** (of Proposition 4.5) Let us begin by defining the connecting homomorphism. Given a class \(\omega\) in \(\pi_n(X, A)\) it can be represented by a map of triples \(H: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)\). This map can be restricted to the top face \(I^n \times \{1\}\) to give a map \(h = H|: (I^{n-1}, \partial I^{n-1}) \rightarrow (A, x_0)\). We set:

\[ \partial: \pi_n(X, A) \rightarrow \pi_{n-1}(A, x_0): \quad [H] \mapsto [h] = [H] \]

We leave it to the reader to check that this defines a group homomorphism or a map of pointed sets depending on the value of \(n\). The naturality of \(\partial\) follows immediately from the definition.

Let us prove that the sequence is exact. Thus, we have to establish exactness at three different positions, one of which we will leave as an exercise. So, we will content ourselves showing exactness at \(\pi_n(A, x_0)\) and at \(\pi_n(X, A)\). So, we have to show that there are four inclusions:
Lecture 4: Relative Homotopy Groups and the Action of the Fundamental Group

(i) \( \text{im}(\partial) \subseteq \ker(i_\ast) \): This inclusion is immediate; given the homotopy class of a map \( H: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \), we have to show that the map \( i \circ h: (I^{n-1}, \partial I^{n-1}) \to (X, x_0) \) is homotopic to the constant map (relative to the boundary). But such a homotopy is given by \( H \) itself.

(ii) \( \ker(i_\ast) \subseteq \text{im}(\partial) \): This follows by definition of the relative homotopy groups and the connecting homomorphism (see the motivational discussion!).

(iii) \( \ker(j_\ast) \subseteq \text{im}(\partial) \): Given an arbitrary \( \alpha \in \pi_n(X, x_0) \) it is easy to see that \( \partial \circ j_\ast \) is by definition represented by the constant map \( \kappa_{x_0}: I^{n-1} \to X \).

(iv) \( \ker(\partial) \subseteq \text{im}(j_\ast) \): Let us consider a map \( H: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \) which lies in the kernel of \( \partial \). By definition this means that the restriction \( h = H|_I: (I^{n-1} \times \{1\}, \partial I^{n-1} \times \{1\}) \to (A, x_0) \) is homotopic to the constant map \( \kappa_{x_0} \) relative to the boundary. Choose an arbitrary such homotopy \( H'\colon h \simeq \kappa_{x_0} \) (rel \( x_0 \)). Then we obtain a map \( H''\colon J^n = I^n \times \{0\} \cup \partial I^n \times I \to X \) which is \( H \) on \( I^n \times \{0\} \), the homotopy \( H' \) on \( I^{n-1} \times \{1\} \times I \) and which takes the constant value \( x_0 \) on the rest of \( \partial I^n \times I \).1 An application of Lemma 4.7 gives us a map \( K: I^{n+1} \to X \) which restricts to \( H'' \) along \( J^n \subseteq I^{n+1} \). By construction, \( K \) is a homotopy of maps of triples \( (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \) from \( H \) to \( K(-, 1): (I^n, \partial I^n) \to (X, x_0) \). Thus, we have \([H] = j_\ast([K(-, 1)])\) as intended.

Exercise 4.8. Conclude the proof of Proposition 4.5 by showing that the sequence is exact at \( \pi_n(X, x_0) \).

Corollary 4.9.

(i) Given a pointed pair of spaces \( (X, A, x_0) \) such that there is a pointed homotopy equivalence \( X \simeq \ast \) then there are isomorphisms \( \pi_n(X, A) \cong \pi_{n-1}(A) \), \( n \geq 1 \).

(ii) Let \( i\colon (A, x_0) \to (X, x_0) \) be the inclusion of a retract, i.e., we have \( r \circ i = \id \) for some pointed map \( r\colon (X, x_0) \to (A, x_0) \). Then there are split short exact sequences

\[
1 \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A) \to 1, \quad n \geq 1,
\]

i.e., short exact sequences such that \( \pi_n(A, x_0) \to \pi_n(X, x_0) \) admits a retraction.

We can apply the first part to the special case of the reduced cone \( CA \) of a pointed space \((A, \ast)\). The reduced cone comes naturally with an inclusion \((A, \ast) \to (CA, \ast)\) so that we have a pointed pair \((CA, A, \ast)\). By the corollary, the connecting homomorphism \( \partial\colon \pi_{n+1}(CA, A) \to \pi_n(A, \ast) \) is an isomorphism. We can combine this with the map induced by the quotient map \( q\colon (CA, A) \to (\Sigma A, \ast) \) in order to obtain the suspension homomorphism:

\[
S\colon \pi_n(A, \ast) \xrightarrow{\delta^{-1}} \pi_{n+1}(CA, A) \xrightarrow{q} \pi_{n+1}(\Sigma A, \ast)
\]

As opposed to the context of singular homology, this suspension homomorphism is not an isomorphism (even not for nice spaces as –say– CW-complexes). However, this map can be iterated

\[1\]In the notation introduced before Lemma 4.7 we thus put the homotopy \( H \) on \( I^{n-1}_{(n, 1)} \times I \) and constant maps \( \kappa_{x_0} \) on \( I^{n-1}_f \times I \subseteq \partial I^n \times I \), \( f \neq (n, 1) \).
and we will later show that the suspension homomorphisms \( S: \pi_{n+k}(\Sigma^k A, \ast) \to \pi_{n+k+1}(\Sigma^{k+1} A, \ast) \) eventually are isomorphisms. Thus, the groups \( \pi_{n+k}(\Sigma^k A, \ast) \) stabilize for large values of \( k \).

3. The action of the fundamental group

We will now turn to the action of the fundamental group on higher homotopy groups. This will also allow us to understand more precisely the difference between \( \pi_n(X, x_0) \) and \([S^n, X]\). To begin with let us collect some basic facts about free homotopies. Given a space \( X \) and a point \( \ast \) where \( \pi \) will also allow us to understand more precisely the difference between \( \pi_n(X, x_0) \) and \([S^n, X]\). To begin with let us collect some basic facts about free homotopies. Given a space \( X \) and a homotopy \( H: S^n \times I \to X \) we obtain a path \( u \) in \( X \) by setting

\[
 u = H(\ast, -): I \to X
\]

where \( \ast \) is the base point of \( S^n \). If \( H \) is a homotopy from \( f \) to \( g \) and if \( u \) is the path of the base point, then this will be denoted by:

\[
 H: f \simeq_u g
\]

The fact that the homotopy relation is an equivalence relation takes the following form if we keep track of the paths of the base point.

Lemma 4.10.  
(1) For every map \( f: S^n \to X \) we have \( f \simeq_{\kappa_{S^n}} f \).

(2) If for two maps \( f, g: S^n \to X \) there is a homotopy \( f \simeq_u g \) then we also have \( g \simeq_{u^{-1}} f \).

(3) Let \( f, g, h: S^n \to X \) be maps such that \( f \simeq_u g \) and \( g \simeq_v h \). Then there is a homotopy \( f \simeq_{u \ast v} h \).

Lemma 4.11. For every map \( f: S^n \to X \) and every path \( u: I \to X \) such that \( u(0) = f(\ast) \) there is a map \( g: S^n \to X \) such that \( f \simeq_u g \).

Proof. Let \( q: I^n \to \partial I^n \simeq S^n \) be the quotient maps. The maps \( f \circ q: I^n \times \{0\} \to X \) and \( u \circ pr: \partial I^n \times I \to I \to X \) together define a map as follows:

\[
 (f \circ q, u \circ pr): J^n = I^n \times \{0\} \cup \partial I^n \times I \to X
\]

It follows from Lemma 4.7 that we can find an extension \( H: I^{n+1} \to X \) as indicated in the diagram. By construction, \( H(-, t): I^n \to X \) takes the constant value \( u(t) \) on the boundary \( \partial I^n \) and hence factors as \( I^n \times I \to S^n \times I \to X \). The induced map \( S^n \times I \to X \) defines a homotopy \( f \simeq_u g \).

Thus \( g \) is obtained from \( f \) by ‘stacking a copy of the path on top of each point of \( \partial I^n \)’ and then choosing a certain reparametrization. In the special case of \( n = 1 \) it is easy to see that this way we obtain \( g = u \ast f \ast u^{-1} \). In the notation of the lemma, we want to show that the assignment

\[
 ([u], [f]) \mapsto [g]
\]

is well-defined.

Lemma 4.12. Let \( f, f_0, f_1, g, g_0, g_1: S^n \to X \) be maps and let \( u, v: I \to X \) be paths in \( X \).

(i) If \( f \simeq_u g \) and \( u \simeq v \) (rel \( \partial I \)) then also \( f \simeq_v g \).

(ii) Let us assume that \( f_0(\ast) = f_1(\ast) = x_0 \) and \( g_0(\ast) = g_1(\ast) = x_1 \). If \( f_0 \simeq f_1 \) (rel \( x_0 \)), \( g_0 \simeq g_1 \) (rel \( x_1 \)) and \( f_0 \simeq_u g_0 \) then also \( f_1 \simeq_u g_1 \).
Proposition 4.14. We recommend that you draw a picture in the case of \( n = 1 \) to see what is happening. Now, let \( H: I^n \times I \to X \) be a homotopy \( f \simeq_u g \) and similarly \( G: I \times I \to X \) a homotopy \( u \simeq v \) (rel \( \partial I \)) which both exist by assumption. From this we construct a new map \( K: J^{n+1} \to X \) as follows. Note that \( J^{n+1} \subseteq \partial I^{n+2} \) can be written as a union of three subspaces (use the Leibniz rule!):

\[
J^{n+1} = I^n \times I \times \{0\} \cup \partial I^n \times I \times I \cup \partial I^n \times I \times I
\]

On the first subspace we take the homotopy \( H \), on the second subspace \( \partial I^n \times I \times I \xrightarrow{pr_0} I \times I \xrightarrow{S} X \), and on the remaining one the constant homotopies of \( f \) and \( g \), i.e., we take:

\[
I^n \times \partial I \times I \xrightarrow{pr_0} I^n \times \partial I \simeq I^n \cup I^n \xrightarrow{(f,g)} X
\]

We leave it to the reader to check that these maps fit together in the sense that they define a map \( K: J^{n+1} \to X \). Now, an application of Lemma 4.7 shows that \( K \) can be extended to a map \( L = K \circ r: I^{n+2} \to J^{n+1} \to X \). By construction it follows that the restriction of \( L \) to \( I^n \times I \times \{1\} \) gives us the desired homotopy \( f \simeq_v g \).

The second claim is now easy. By assumption we have a chain of homotopies:

\[
f_1 \simeq_{\kappa_{x_0}} f_0 \simeq_u g_0 \simeq_{\kappa_{x_1}} \simeq g_1
\]

But since \( \kappa_{x_1} * u * \kappa_{x_0} \simeq u \) (rel \( \partial I \)) we can conclude \( f_1 \simeq_u g_1 \) (by the first part of this lemma). □

Recall that given a space \( X \) we denote its fundamental groupoid by \( \pi_1(X) \). The objects in \( \pi_1(X) \) are the points in \( X \) while morphisms are given by homotopy classes of paths relative to the boundary.

**Corollary 4.13.** Let \( f: (S^n,*) \to (X, x_0) \), let \( u: I \to X \) be a path from \( x_0 \) to \( x_1 \), and let \( f \simeq_u g \) for some \( g: (S^n,*) \to (X, x_1) \). Then the homotopy class \([g] \in \pi_n(X,x_1)\) only depends on the homotopy classes \([f] \in \pi_n(X,x_0)\) and \([u] \in \pi_1(X)\).

**Proof.** Let us assume we were also given \( f \simeq_{\kappa_{x_0}} f' \), \( u \simeq v \) (rel \( \partial I \)), and \( f' \simeq_v g' \). Then in order to show that \( g \simeq_{\kappa_{x_1}} g' \) we observe that:

\[
g \simeq_{u^{-1}} f \simeq_{\kappa_{x_0}} f' \simeq_v g'
\]

But since \( v * \kappa_{x_0} * u^{-1} \simeq_{\kappa_{x_1}} \) (rel \( \partial I \)) we can conclude by Lemma 4.10 and Lemma 4.12. □

Thus, we obtain a well-defined pairing

\[
\pi_n(X,x_0) \times \pi_n(X,x_0) \to \pi_n(X,x_1): ([u],[f]) \mapsto [u][f] = [g]
\]

for \( f \simeq_u g \) as in the notation of Lemma 4.11.

**Proposition 4.14.** Given a space \( X \) then we have a functor \( \pi_n(X,-): \pi_1(X) \to \text{Grp} \) which sends an object \( x_0 \in \pi_1(X) \) to \( \pi_n(X,x_0) \) and a map \([u] \in \pi_1(X,x_0)\) to \([u](-): \pi_n(X,x_0) \to \pi_n(X,x_1)\).

**Proof.** We know already that \( \pi_n(X,x_0) \) is a group for all \( x_0 \in X \) and that we have a well-defined map of sets \([u](-): \pi_n(X,x_0) \to \pi_n(X,x_1)\). To check that the assignment \([u] \mapsto [u](-)\) is compatible with compositions and identities it suffices to recall the definition of this action. In fact, since it was obtained from 'stacking copies of \( u \) on top of \( \partial I^n \) it is easy to see that this is true. It remains to show that the maps \([u](-): \pi_n(X,x_0) \to \pi_n(X,x_1)\) are group homomorphisms. But this is left as an exercise. □
Exercise 4.15. Given a path \( u: I \to X \) with \( u(0) = x_0 \) and \( u(1) = x_1 \) show that \([f] \mapsto [u][f]\) defines a group homomorphism

\[
[u](-): \pi_n(X, x_0) \to \pi_n(X, x_1).
\]

Thus, we have isomorphisms \( \pi_n(X, x_0) \cong \pi_n(X, x_1) \) whenever \( x_0, x_1 \in X \) lie in the same path-component. Note that such an isomorphism is, in general, not canonical, since it depends on the choice of a homotopy class of paths from \( x_0 \) to \( x_1 \). However, if \( \pi_1(X, x_0) \cong 1 \) then there is only a unique such homotopy class so that the identification \( \pi_n(X, x_0) \cong \pi_n(X, x_1) \) is canonical.

Corollary 4.16. Given a pointed space \((X, x_0)\) then there is an action of \( \pi_1(X, x_0) \) on \( \pi_n(X, x_0) \). For \( n = 1 \) this specializes to the conjugation action, i.e., we have:

\[
[u][f] = [u][f][u]^{-1}, \quad [u], [f] \in \pi_1(X, x_0)
\]

Proof. Since we have a functor \( \pi_n(X, -): \pi(X) \to \text{Grp} \), it is completely formal that we get an action of \( \pi_1(X, x_0) \) on \( \pi_n(X, x_0) \). In the context of Lemma 4.11, we already observed that our construction sends \((u, f)\) to \( u \ast f \ast u^{-1} \). Thus, at the level of homotopy classes we obtain the conjugation. \( \square \)

Instead of using the actual construction of Lemma 4.11 to deduce this corollary, we can also argue using the essential uniqueness of the construction (Corollary 4.13): we just have to observe that there is a homotopy:

\[ f \simeq_u u \ast f \ast u^{-1} \]

Whenever we have a group acting on a set we can pass to the set of orbits. In the case of the action of the fundamental group on higher homotopy groups we obtain the following convenient result.

Corollary 4.17. Let \( X \) be a path-connected space and let \( x_0 \in X \). Then the forgetful map

\[
\pi_n(X, x_0) = [(S^n, \ast), (X, x_0)] \to [S^n, X]
\]

exhibits \([S^n, X]\) as the set of orbits of the action of \( \pi_1(X, x_0) \) on \( \pi_n(X, x_0) \).

Exercise 4.18. Give a proof of this corollary, i.e., show that the forgetful map is surjective and that two elements \([f]\) and \([g]\) have the same image if and only if there is a loop \( u \) at \( x_0 \) such that \([u][f] = [g]\).
LECTURE 5: FIBRATIONS AND HOMOTOPY FIBERS

In this lecture we will introduce two important classes of maps of spaces, namely the Hurewicz fibrations and the more general Serre fibrations, which are both obtained by imposing certain homotopy lifting properties. We will see that up to homotopy equivalence every map is a Hurewicz fibration. Moreover, associated to a Serre fibration we obtain a long exact sequence in homotopy which relates the homotopy groups of the fibre, the total space, and the base space. This sequence specializes to the long exact sequence of a pair which we already discussed in the previous lecture.

1. Fibrations

Definition 5.1.

(i) A map \( p: E \to X \) of spaces is said to have the right lifting property (RLP) with respect to a map \( i: A \to B \) if for any two maps \( f: A \to E \) and \( g: B \to X \) with \( pf = gi \), there exists a map \( h: B \to E \) with \( ph = g \) and \( hi = f \):

\[
\begin{array}{ccc}
A & \xrightarrow{f} & E \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{h} & E \\
& & \downarrow{g} \\
& & X
\end{array}
\]

(So \( h \) at the same time ‘extends’ \( f \) and ‘lifts’ \( g \).

(ii) A map \( p: E \to X \) of spaces is a Serre fibration if it has the RLP with respect to all inclusions of the form

\[ I^n \times \{0\} \to I^n \times I = I^{n+1}, \quad n \geq 0, \]

and a Hurewicz fibration if it has the RLP with respect to all maps of the form

\[ A \times \{0\} \to A \times I \]

for any space \( A \). (So evidently, every Hurewicz fibration is a Serre fibration.)

(iii) If \( p: E \to X \) is a map of spaces (but typically one of the two kinds of fibrations) and \( x \in X \), then \( p^{-1}(x) \subseteq E \) is called the fiber of \( p \) over \( x \). If \( x = x_0 \) is a base point specified earlier, we just say the fiber of \( p \) for the fiber over \( x_0 \).

Thus, Hurewicz fibrations are those maps \( p: E \to X \) which have the homotopy lifting property with respect to all spaces: given a homotopy \( H: A \times I \to X \) of maps with target \( X \) and a lift \( G_0: A \to E \) of \( H_0 = H(-,0): A \to X \) against the fibration \( p: E \to X \) then this partial lift can be extended to a lift of the entire homotopy \( G: A \times I \to E \), i.e., \( G \) satisfies \( pG = H \) and \( Gi = G_0 \):

\[
\begin{array}{ccc}
A \times \{0\} & \xrightarrow{G_0} & E \\
\downarrow{i} & \Downarrow{G} & \downarrow{p} \\
A \times I & \xrightarrow{H} & X
\end{array}
\]
By definition, the class of Serre fibrations is given by those maps which have the homotopy lifting property with respect to all cubes \( I^n \).

**Example 5.2.**

(i) Any projection \( X \times F \to X \) is a Hurewicz fibration, as you can easily check.

(ii) The evaluation map

\[
\epsilon = (\epsilon_0, \epsilon_1) : X^I \to X \times X
\]

at both end points is a Hurewicz fibration. Indeed, suppose we are given a commutative square

\[
\begin{array}{c}
A \times \{0\} \xrightarrow{f} X^I \\
i \downarrow \quad \downarrow h \\
A \times I \xrightarrow{g} X \times X
\end{array}
\]

Or equivalently, we are given a map

\[
\phi : (A \times \{0\} \times I) \cup (A \times I \times \{0,1\}) \to X
\]

which we wish to extend to \( A \times I \times I \). But \((\{0\} \times I) \cup (I \times \{0,1\}) = J^1 \subseteq I^2\) is a retract of \( I^2 \), and hence so is \( A \times J^1 \subseteq A \times I^2 \). Therefore we can simply precompose \( \phi \) with the retraction \( r : A \times I^2 \to A \times J^1 \) to find the required extension.

(iii) Let \( p : E \to X \) and \( f : X' \to X \) be arbitrary maps. Form the fibered product or pullback

\[
E \times_X X' = \{ (e, x') \mid p(e) = f(x') \}
\]

topologized as a subspace of the product \( E \times X' \). Then if \( E \to X \) is a Hurewicz (or Serre) fibration, so is the induced projection \( E \times_X X' \to X' \). This follows easily from the universal property of the pullback (see Exercise 1).

(iv) If \( E \to D \to X \) are two Hurewicz (or Serre) fibrations, then so is their composition \( E \to X \) (see Exercise 2).

(v) Let \( (X, x_0) \) be a pointed space. The path space \( P(X) \) (or more precisely, \( P(X, x_0) \) if necessary) is the subspace of \( X^I \) (always with the compact-open topology) given by paths \( \alpha \) with \( \alpha(0) = x_0 \). The map \( \epsilon_1 : P(X) \to X \) given by evaluation at 1 is a Hurewicz fibration. This follows by combining the previous examples (ii) and (iii).

(vi) If \( f : Y \to X \) is any map, the mapping fibration of \( f \) is the map

\[
p : P(f) \to X
\]

constructed as follows. The space \( P(f) \) is the fibered product

\[
P(f) = X^I \times_X Y = \{ (\alpha, y) \mid \alpha(1) = f(y) \}
\]

and the map \( p \) is given by \( p(\alpha, y) = \alpha(0) \). We claim that \( p \) is a Hurewicz fibration. Indeed, suppose we are given a commutative diagram

\[
\begin{array}{ccc}
A \times \{0\} & \xrightarrow{u} & P(f) \\
\downarrow i & & \downarrow p \\
A \times I & \xrightarrow{v} & X
\end{array}
\]
Denoting by $\pi_1: P(f) \to X'$ and $\pi_2: P(f) \to Y$ the two projection maps belonging to the pullback $P(f)$, we can first extend $\pi_2 \circ u$ as in:

$$
A \times \{0\} \overset{\pi_2 \circ u}{\longrightarrow} Y \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Thus, the fiber ‘sits in’ the homotopy fiber while the homotopy fiber can be thought of as a ‘relaxed’ version of the fiber: the condition imposed on a point \( y \in Y \) to lie in the fiber over \( x \) is that it has to be mapped to \( x \) by \( f \), i.e., \( f(y) = x \), while a point of the homotopy fiber is a pair \((\alpha, y)\) consisting of \( y \in Y \) together with a path \( \alpha \) in \( X \) ‘witnessing’ that \( y \) ‘lies in the fiber up to homotopy’.

2. The long exact sequence associated to a fibration

So far, all our examples are examples of Hurewicz fibrations. However, we will see in the next lecture that the weaker property of being a Serre fibration is a \textit{local property}, and hence that all fiber bundles are examples of Serre fibrations. Moreover, this weaker notion suffices to establish the following theorem.

**Theorem 5.4.** (The long exact sequence of a Serre fibration)

Let \( p : (E, e_0) \to (X, x_0) \) be a map of pointed spaces with \( i : (F, e_0) \to (E, e_0) \) being the fiber. Suppose that \( p \) is a Serre fibration. Then there is a long exact sequence of the form:

\[
\cdots \to \pi_{n+1}(X, x_0) \xrightarrow{\delta} \pi_n(E, e_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{\delta} \pi_{n-1}(X, x_0) \cdots \xrightarrow{\delta} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(X, x_0)
\]

The ‘connecting homomorphism’ \( \delta \) will be constructed explicitly in the proof. Before turning to the proof, let us deduce an immediate corollary. By considering the homotopy fiber \( H_f \) instead of the actual fiber, we see that we can obtain a long exact sequence for an arbitrary map \( f \) of pointed spaces.

**Corollary 5.5.** Let \( f : (Y, y_0) \to (X, x_0) \) be a map of pointed spaces and let \( H_f \) be its homotopy fiber. Then there is a long exact sequence of the form:

\[
\cdots \to \pi_{n+1}(X, x_0) \xrightarrow{\delta} \pi_n(Y, y_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{\delta} \pi_{n-1}(X, x_0) \cdots \xrightarrow{\delta} \pi_0(Y, y_0) \xrightarrow{i_*} \pi_0(X, x_0)
\]

**Proof.** Apply Theorem 5.4 to the mapping fibration (Example 5.2(vi)) and use Exercise 3.

Before entering in the proof of the theorem, we recall from the previous lecture the definition of the subspace \( J^n \subseteq I^{n+1} \):

\[
J^n = (I^n \times \{0\}) \cup (\partial I^n \times I) \subseteq \partial I^{n+1} \subseteq I^{n+1}.
\]

Note that, by ‘flattening’ the sides of the cube, one can construct a homeomorphism of pairs

\[
(I^{n+1}, J^n) \xrightarrow{\cong} (I^{n+1}, I^n \times \{0\}).
\]

Thus, any Serre fibration also has the RLP with respect to the inclusion \( J^n \subseteq I^{n+1} \). We will use this repeatedly in the proof.

**Proof of Theorem 5.4.** The main part of the proof consists in the construction of the operation \( \delta \).

Let \( \alpha : (I^n, \partial I^n) \to (X, x_0) \) represent an element of \( \pi_n(X, x_0) = \pi_n(X) \). Let \( \bar{c}_0 : J^{n-1} \to E \) be the constant map with value \( c_0 \). Then the square

\[
\begin{array}{ccc}
J^{n-1} & \xrightarrow{\bar{c}_0} & E \\
\downarrow & & \downarrow p \\
I^n & \xrightarrow{\alpha} & X
\end{array}
\]

commutes, so by the definition of a Serre fibration we find a diagonal \( \beta \). Then \( \delta[\alpha] \) is the element of \( \pi_{n-1}(F) \) represented by the map

\[
\beta(-, 1) : I^{n-1} \to F, \quad t \mapsto \beta(t, 1).
\]
Note that this indeed represents an element of $\pi_{n-1}(F)$, because the boundary of $I^{n-1} \times \{1\}$ is contained in $J^{n-1}$, and $\beta$ maps the top face $I^{n-1} \times \{1\}$ into $F$ since $p \circ \beta = \alpha$ maps it to $x_0$.

The first thing to check is that $\delta$ is well defined on homotopy classes. Suppose $[\alpha_0] = [\alpha_1]$, as witnessed by a homotopy $h : I \times I \to X$ from $\alpha_0$ to $\alpha_1$. Suppose also that we have chosen liftings $\beta_0$ and $\beta_1$ of $\alpha_0$ and $\alpha_1$ as above. Then we can define a map $k$ making the solid square

\[
\begin{array}{ccc}
J^n & \xrightarrow{k} & E \\
\downarrow & & \downarrow p \\
I^n \times I & \xrightarrow{\beta} & X \\
\end{array}
\]

commute. Here $J^n$ is the union of all the faces of $I^{n+1}$ except $\{t_n = 1\}$. (It is like $J^n$ except that we have interchanged the roles of $t_n$ and $t_{n+1}$.) On $I^n \times \{0\}$ and $I^n \times \{1\}$ the map $k$ is defined to be $\beta_0$ and $\beta_1$ respectively. On the faces $\{t_i = 0\}$, $\{t_i = 1\}$ ($i < n$) and $\{t_n = 0\}$ the map $k$ has constant value $e_0$. Now a diagonal $l$ restricted to $I^{n-1} \times \{1\} \times I$ gives a homotopy from $\beta_0(-,1)$ to $\beta_1(-,1)$, and lies entirely in the fiber over $x_0$ because $h$ is a homotopy relative to $\partial I^n$. This proves that $\beta_0(-,1)$ and $\beta_1(-,1)$ define the same element of $\pi_{n-1}(F)$. It also proves that $\delta[\alpha]$ thus defined does not depend on the choice of the filling $\beta$ (Why?).

With these details about $\delta$ being well-defined out of the way, it is quite easy to prove that the sequence of the theorem is an exact sequence of pointed sets. (We write ‘pointed sets’ here because we haven’t proved yet that $\delta$ is a homomorphism of groups for $n > 1$. We leave this to you as Exercise 5.7.)

Exactness at $\pi_n(E)$. Clearly $p_* \circ i_* = 0$ because $p \circ i$ is constant, so $\text{im}(i_*) \subseteq \ker(p_*)$. For the reverse inclusion, suppose $\alpha : I^n \to E$ represents an element of $\pi_n(E)$ with $p_*[\alpha] = [p \circ \alpha] = 0$. Let $h : I^n \times I \to X$ be a homotopy rel $\partial I^n$ from $p\alpha$ to the constant map on $x_0$. Choose a lift $l$ in

\[
\begin{array}{ccc}
J^n & \xrightarrow{k} & E \\
\downarrow & & \downarrow p \\
I^n \times I & \xrightarrow{\gamma} & X \\
\end{array}
\]

where $k|_{I^n \times \{0\}} = \alpha$ and $k$ is constant $e_0$ on the other faces. Then $\gamma = l|_{I^n \times \{1\}}$ maps entirely into $F$, so represents an element $[\gamma] \in \pi_n(F)$ with $i_*[\gamma] = [i\gamma] = [\alpha]$ (by the homotopy $l$).

Exactness at $\pi_n(X)$. If $\beta : I^n \to E$ represents an element of $\pi_n(E)$ then for $\alpha = p \circ \beta$ we can take the same $\beta$ as the diagonal filling in the construction of $\delta[\alpha]$. So $\delta p_*[\beta] = [p \circ \beta]_{I^{n-1} \times \{1\}}$ which is constant $e_0$. Thus $\delta \circ p_* = 0$, or $\text{im}(p_*) \subseteq \ker(\delta)$. For the converse inclusion, suppose $\alpha : I^n \to X$ represents an element of $\pi_n(X)$ with $\delta[\alpha] = 0$. Then for a lift $\beta$ as in

\[
\begin{array}{ccc}
J^{n-1} & \xrightarrow{\tilde{\delta}} & E \\
\downarrow & & \downarrow p \\
I^n & \xrightarrow{\beta} & X \\
\end{array}
\]

we have that $\beta(-,1)$ is homotopic to the constant map by a homotopy $h$ relative to $\partial I^{n-1}$ which maps into the fiber $F$. But then, stacking this homotopy $h$ on top of $\beta$ (i.e., by forming $h \circ_n \beta$), we obtain a map representing an element $\beta'$ of $\pi_n(E)$. The image $p \circ \beta'$ is obviously homotopic to $p \circ \beta = \alpha$ because $p \circ h$ is constant, showing that $[\alpha]$ lies in the image of $p_*$. 
Exactness at $\pi_{n-1}(F)$. For $\alpha: I^n \to X$ representing an element of $\pi_n(X)$, the map $\beta$ in the construction of $\delta[\alpha] = [\beta(\cdot,1)]$ shows that $\beta(\cdot,1) = \bar{e}_0$ in $E$, so $i_*[\alpha] = 0$. For the other inclusion, suppose $\gamma: I^{n-1} \to F$ represents an element of $\pi_{n-1}(F)$ with $i_*[\gamma] = 0$, as represented by a homotopy $h: I^{n-1} \times I \to E$ with $h(\cdot,1) = \gamma$ and $h(\cdot,0) = \bar{e}_0$. Then $\alpha = p \circ h$ represents an element of $\pi_n(X)$, and in the construction of $\delta[\alpha]$ we can choose the diagonal filling $\beta$ to be identical to $h$, in which case $\delta[\alpha]$ is represented by $\gamma$. This shows that $\ker(i_*) \subseteq \text{im}($, and completes the proof of the theorem.

Exercise 5.6. Show that the long exact sequence of a pointed pair $(X,A)$, constructed in the previous lecture, can be obtained from this long exact sequence, by considering the mapping fibration of the inclusion $A \to X$ (see also the last exercise sheet).

Exercise 5.7. Prove that the connecting homomorphism $\delta: \pi_n(X,x_0) \to \pi_{n-1}(F,\bar{e}_0)$ is a homomorphism of groups for $n \geq 2$.

Exercise 5.8. Let $p: E \to X$ be a Hurewicz fibration, and let $\alpha: I \to X$ be a path from $x$ to $y$. Use the lifting property of $E \to X$ with respect to $p^{-1}(x) \times \{0\} \to p^{-1}(x) \times I$ to show that $\alpha$ induces a map $\alpha_*: p^{-1}(x) \to p^{-1}(y)$. Show that the homotopy class of $\alpha_*$ only depends on the homotopy class of $\alpha$, and that this construction in fact defines a functor on the fundamental groupoid, $\pi(X) \to \text{Ho}($Top$)$.

This last exercise shows in particular that the homotopy type of the fiber of a Hurewicz fibration is constant on path components. More precisely, if $p: E \to X$ is a Hurewicz fibration, then any path $\alpha: I \to X$ induces a homotopy equivalence between the fiber over $\alpha(0)$ and the fiber over $\alpha(1)$. 

LECTURE 6: FIBER BUNDLES

In this section we will introduce the interesting class of fibrations given by fiber bundles. Fiber bundles play an important role in many geometric contexts. For example, the Grassmannian varieties and certain fiber bundles associated to Stiefel varieties are central in the classification of vector bundles over (nice) spaces. The fact that fiber bundles are examples of Serre fibrations follows from Theorem 6.11 which states that being a Serre fibration is a local property.

1. Fiber bundles and principal bundles

Definition 6.1. A fiber bundle with fiber \( F \) is a map \( p: E \to X \) with the following property: every point \( x \in X \) has a neighborhood \( U \subseteq X \) for which there is a homeomorphism \( \phi_U: U \times F \cong p^{-1}(U) \) such that the following diagram commutes in which \( \pi_1: U \times F \to U \) is the projection on the first factor:

\[
\begin{array}{ccc}
U \times F & \xrightarrow{\phi_U} & p^{-1}(U) \\
\downarrow \pi_1 & & \downarrow p \\
U & \xrightarrow{p} & X
\end{array}
\]

Remark 6.2. The projection \( X \times F \to X \) is an example of a fiber bundle: it is called the trivial bundle over \( X \) with fiber \( F \). By definition, a fiber bundle is a map which is ‘locally’ homeomorphic to a trivial bundle. The homeomorphism \( \phi_U \) in the definition is a local trivialization of the bundle, or a trivialization over \( U \).

Let us begin with an interesting subclass. A fiber bundle whose fiber \( F \) is a discrete space is (by definition) a covering projection (with fiber \( F \)). For example, the exponential map \( \mathbb{R} \to S^1 \) is a covering projection with fiber \( \mathbb{Z} \). Suppose \( X \) is a space which is path-connected and locally simply connected (in fact, the weaker condition of being semi-locally simply connected would be enough for the following construction). Let \( \tilde{X} \) be the space of homotopy classes (relative endpoints) of paths in \( X \) which begin at a given base point \( x_0 \). We can equip \( \tilde{X} \) with the quotient topology with respect to the map \( P(X, x_0) \to \tilde{X} \). The evaluation \( \epsilon_1: P(X, x_0) \to X \) induces a well-defined map \( \tilde{X} \to X \). One can show that \( \tilde{X} \to X \) is a covering projection. (It is called the universal covering projection. A later exercise will explain this terminology.)

Let \( p: E \to X \) be a fiber bundle with fiber \( F \). If \( f: X' \to X \) is any map, then the projection

\[
f^*(p): X' \times_X E \to X'
\]

is again a fiber bundle with fiber \( F \) (see Exercise 1 of sheet 6).

We will need the following definitions.

Definition 6.3. Let \( f: Y \to X \) be an arbitrary map.

(i) A section of \( f \) over an open set \( U \subseteq X \) is a map \( s: U \to Y \) such that \( f \circ s = id_U \).

(ii) The map \( f: Y \to X \) has enough local sections if every points of \( X \) has an open neighborhood on which some local section of \( f \) exists.
Thus, by the very definition, every fiber bundle has enough local sections. And if you know a
bit of differential topology, you’ll know that any surjective submersion between smooth manifolds
has enough local sections.

Many interesting examples of fiber bundle show up in the context of nice group actions. For this
purpose, let us formalize this notion.

**Definition 6.4.** Let $G$ be a topological group and let $E$ be a space. A (right) action of $G$ on $E$
is a map 

$$\mu: E \times G \to E: (e, g) \mapsto \mu(e, g) = e \cdot g$$

such that the following identities hold:

$$e \cdot 1 = e \quad \text{and} \quad e \cdot (gh) = (e \cdot g) \cdot h, \quad e \in E, \ g, h \in G.$$ 

If $E$ also comes with a map $p: E \to X$ such that $p(e \cdot g) = p(e)$ for all $e$ and $g$, then the action of $G$
restricts to an action on each fiber of $p$, and one also says that the action is fiberwise.

Given a space $E$ with a right action by $G$, then there is an induced equivalence relation $\sim$ on $E$ defined by

$$e \sim e' \iff e \cdot g = e' \quad \text{for some} \quad g \in G.$$ 

The quotient space $E/\sim$ is called the orbit space of the action, and is usually denoted $E/G$. The equivalence classes are called the orbits of the action. They are the fibers of the quotient map

$$\pi: E \to E/G.$$ 

**Definition 6.5.** Let $G$ be a topological group. A principal $G$-bundle is a map $p: E \to B$ together
with a fiberwise action of $G$ on $E$, with the property that:

(i) The map $\phi: E \times G \to E \times B: (e, g) \mapsto (e, e \cdot g)$ is a homeomorphism.

(ii) The map $p: E \to B$ has enough local sections.

**Proposition 6.6.** Any principal $G$-bundle is a fiber bundle with fiber $G$.

**Proof.** Write

$$\delta = \pi_2 \circ \phi^{-1}: E \times G \to E \times G \to G$$

for the difference map, characterized by the identity

$$e \cdot \delta(e, e') = e'$$

for any $(e, e') \in E \otimes_B E$. If $b \in B$ and $b \in U \subseteq B$ is a neighborhood on which a local section $s: U \to E$ exists, then the map

$$U \times G \to p^{-1}(U): (x, g) \mapsto s(x) \cdot g$$

is a homeomorphism, with inverse given by $e \mapsto (p(e), \delta(s(p(e)), e))$. \hfill \Box

An important source of principal bundles comes from the construction of homogeneous spaces.
Let $G$ be a topological group, and suppose that $G$ is compact and Hausdorff. Let $H$ be a closed subgroup of $G$, and let $G/H$ be the space of left cosets $gH$. Then the projection $\pi: G \to G/H$
satisfies the first condition in the definition of principal bundles, because the map

$$\phi: G \times H \to G \times (G/H) G: (g, h) \mapsto (g, gh)$$

is easily seen to be a continuous bijection, and hence it is a homeomorphism by the compact-
Hausdorff assumption. So, we conclude that if $G \to G/H$ has enough local sections, then it is
a principal $H$-bundle. (For those who know Lie groups: if $G$ is a compact Lie group and $H$ is a
closed subgroup, then $G/H$ is a manifold and $G \to G/H$ is a submersion, hence has enough local sections.)

**Remark 6.7.** In case you have to prove by hand that $\pi: G \to G/H$ has enough local sections, it is useful to observe that it suffices to find a local section on a neighborhood $V$ of $\pi(1)$ where $1 \in G$ is the unit, so $\pi(1) = H \in G/H$. Because if $s: V \to G$ is such a section, then for another coset $gH$, the open set $gV$ is a neighborhood of $gH$ in $G/H$, and $\tilde{s}: gV \to G$ defined $\tilde{s}(\xi) = gs(g^{-1}\xi)$ is a local section on $gV$.

**Remark 6.8.** A related construction yields for two closed subgroups $K \subseteq H \subseteq G$ a map $G/K \to G/H$ which gives us a fiber bundle with fiber $H/K$ under certain assumptions. (See Exercise 5 of sheet 6)

2. **Stiefel varieties and Grassmann varieties**

We will now consider some classical and important special cases of these general constructions for groups, namely the cases of Stiefel and Grassmann varieties. We begin by the Stiefel varieties. Consider the vector space $\mathbb{R}^n$ with its standard basis $(e_1, \ldots, e_n)$. A $k$-frame in $\mathbb{R}^n$ (or more explicitly, an orthonormal $k$-frame) is a $k$-tuple of vectors in $\mathbb{R}^n$, 

$$(v_1, \ldots, v_k)$$

with $\langle v_i, v_j \rangle = \delta_{ij}$. Thus, $v_1, \ldots, v_k$ form an orthonormal basis for a $k$-dimensional subspace $\text{sp}(v_1, \ldots, v_k) \subseteq \mathbb{R}^n$. We can topologize this space of $k$-frames as a subspace of $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ ($k$ times). It is a closed and bounded subspace, hence it is compact. This space is usually denoted $V_{n,k}$ and called the *Stiefel variety*. (It has a well-defined dimension: what is it?) Note that $V_{n,1} = S^{n-1}$ is a sphere. We claim that $V_{n,k}$ is a homogeneous space, i.e., a space of the form $G/H$ as just discussed. To see this, take for $G$ the group $O(n)$ of orthogonal transformations of $\mathbb{R}^n$. We can think of the elements of $O(n)$ as orthogonal $n \times n$ matrices, or as $n$-tuples of vectors in $\mathbb{R}^n$, 

$$(v_1, \ldots, v_n)$$

(the column vectors of the matrix) which form an orthonormal basis in $\mathbb{R}^n$. Thus, there is an evident projection

$$\pi: O(n) \to V_{n,k}$$

which just remembers the first $k$ vectors. The group $O(n-k)$ can be viewed as a closed subgroup of $O(n)$, using the group homomorphism

$$O(n-k) \to O(n): A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$$

where $I = I_k$ is the $k \times k$ unit matrix. One easily checks (Exercise!) that the projection induces a homeomorphism

$$O(n)/O(n-k) \cong V_{n,k}.$$ 

Note that it is again enough to show that we have a continuous bijection since the spaces under consideration are compact Hausdorff. Thus, to see that $\pi: O(n) \to V_{n,k}$ is a principal bundle, it suffices to check that there are enough local sections. This can easily be done explicitly, using the Gram-Schmidt algorithm for transforming a basis into an orthonormal one. Indeed, as we said
above, it is enough to find a local section on a neighborhood of $\pi(1) = \pi(e_1, \ldots, e_n) = (e_1, \ldots, e_k)$. Let

$$U = \{(v_1, \ldots, v_k) | v_1, \ldots, v_k, e_{k+1}, \ldots, e_n \text{ are linearly independent}\}$$

and let $s(v_1, \ldots, v_k)$ be the result of applying the Gram-Schmidt to the basis $v_1, \ldots, v_k, e_{k+1}, \ldots, e_n$. (This leaves $v_1, \ldots, v_k$ unchanged, changes $e_{k+1}$ into $e_{k+1} - \sum (v_i, e_{k+1}) v_i$ divided by its length, and so on.)

Let us observe that this construction of the Stiefel varieties also shows that they fit into a tower

$$O(n) = V_{n,n} \to V_{n,n-1} \to \ldots \to V_{n,k} \to V_{n,k-1} \to \ldots \to V_{n,1} \cong S^{n-1}$$

in which each map $V_{n,k} \to V_{n,k-1}$ is a principal bundle with fiber:

$$O(n-k+1)/O(n-k) \cong V_{n-k+1,1} \cong S^{n-k}$$

From these Stiefel varieties we can now construct the Grassmann varieties. In fact, the group $O(k)$ obviously acts on the Stiefel variety $V_{n,k}$ of $k$-frames in $\mathbb{R}^n$. The orbit space of this action is called the Grassmann variety, and denoted

$$G_{n,k} = V_{n,k}/O(k).$$

The orbit of a $k$-frame $(v_1, \ldots, v_k)$ only remembers the subspace $W$ spanned by $(v_1, \ldots, v_k)$, because any two orthogonal bases for $W$ can be related by acting by an element of $O(k)$. Thus, $G_{n,k}$ is the space of $k$-dimensional subspaces of $\mathbb{R}^n$. Since $V_{n,k} = O(n)/O(n-k)$, the Grassmann variety is itself a homogeneous space

$$G_{n,k} \cong O(n)/(O(k) \times O(n-k))$$

where $O(k)$ and $O(k) \times O(n-k)$ are viewed as the subgroups of matrices of the forms

$$\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$$

respectively. The quotient map

$$q: O(n) \to G_{n,k}$$

is again a principal bundle (with fiber $O(k) \times O(n-k)$), because $q$ again has enough local sections. Indeed, it suffices to construct a local section on a neighborhood of $q(I)$. As a $k$-dimensional subspace of $\mathbb{R}^n$ this is $\mathbb{R}^k \times \{0\}$. Let

$$U = \{W \subseteq \mathbb{R}^n | W \oplus \mathbb{R}^{n-k} = \mathbb{R}^n\}$$

be the subspace of complements of the subspace $\mathbb{R}^{n-k} \subseteq \mathbb{R}^n$ spanned by $e_{k+1}, \ldots, e_n$, and define a section $s$ on $U$ as follows: write $w_i$ for the projection of $e_i$ on $W$, $1 \leq i \leq k$, i.e.,

$$e_i = w_i + \sum_{j>k} \lambda_{ij} e_j.$$

Then $(w_1, \ldots, w_k, e_{k+1}, \ldots, e_n)$ still span all of $\mathbb{R}^n$, and we can transform this into an orthonormal basis by Gram-Schmidt, the result of which defines $s(W)$.

It follows that $V_{n,k} \to G_{n,k}$ also has enough local sections (why?), so this is a principal bundle too (for the group $O(k)$). Summarizing, we have a diagram of three principal bundles:

$$\begin{align*}
&O(n) \\
&\downarrow \\
&V_{n,k} & \longrightarrow & G_{n,k}
\end{align*}$$
constructed as
\[ O(n) \to O(n)/O(n - k) \to O(n)/(O(k) \times O(n - k)). \]

3. Fiber bundles are fibrations

The relation of the previous considerations to the previous lecture is given by the following result.

**Theorem 6.9.** A fiber bundle is a Serre fibration.

Before proving this theorem, we draw some immediate consequences by applying the long exact sequence of homotopy groups associated to a Serre fibration to our examples of fiber bundles. More applications of this kind can be found in the exercises.

**Application 6.10.**

(i) Let \( p: E \to B \) be a covering projection, let \( e_0 \in E \) and let \( b_0 = p(e_0) \). If we denote the fiber by \( F \) (a discrete space), then we have pointed maps
\[ (F, e_0) \to (E, e_0) \to (B, b_0). \]
Then \( p_*: \pi_i(E, e_0) \to \pi_i(B, b_0) \) is an isomorphism for all \( i > 0 \). Moreover, if \( E \) is connected then there is short exact sequence
\[ 0 \to \pi_1(E) \to \pi_1(B) \to F \to 0 \]
where we have omitted base points from notation, and where we view \( F \) as a pointed set \((F, e_0)\). Thus, for the covering \( \mathbb{R} \to S^1 \) this gives us \( \pi_i(S^1) \cong 0 \) for \( i > 1 \), since \( \mathbb{R} \) is contractible.

More generally, for the universal covering projection \( \tilde{X} \to X \) with fiber \( \pi_1(X, x_0) \) we have \( \pi_i(\tilde{X}) \cong \pi_i(X) \) for \( i > 1 \) and \( \pi_1(\tilde{X}) \cong 0 \). These statements all follow by applying the long exact sequence of a Serre fibration.

(ii) In the second lecture we stated that \( \pi_i(S^n) \cong 0 \) for \( i < n \) (a fact that can easily be proved using a bit of differential topology, but which we haven’t given an independent proof yet). Using this, we can analyze the long exact sequence associated to the fiber bundle
\[ O(n) \to V_{n,1} \cong S^{n-1} \]
with fiber \( O(n - 1) \), to conclude that the map
\[ \pi_i(O(n - 1)) \to \pi_i(O(n)) \]
induced by the inclusion (always with the unit of the group as the base point) is an isomorphism for \( i + 1 < n - 1 \) and a surjection for \( i < n - 1 \). Writing \( O(n - k) \to O(n) \) as a composition
\[ O(n - k) \to O(n - k + 1) \to \ldots \to O(n - 1) \to O(n), \]
we find that
\[ \pi_i(O(n - k)) \to \pi_i(O(n)) \]
is an isomorphism if \( i + 1 < n - k \) and is surjective if \( i < n - k \). Feeding this back in the long exact sequence for the fiber bundle
\[ O(n) \to O(n)/O(n - k) \cong V_{n,k}, \]
we conclude that
\[ \pi_i(V_{n,k}) \cong 0, \quad i < n - k. \]
Now back to the proof of Theorem 6.9. Instead of proving this theorem, we will prove a slightly more general result (Theorem 6.11), which can informally be phrased by saying that ‘being a Serre fibration is a local property’. Theorem 6.9 immediately follows from this result and the fact that trivial fibrations are Serre fibrations.

**Theorem 6.11.** Let \( p: E \to B \) be a map with the property that every point \( b \in B \) has a neighborhood \( U \subseteq B \) such that the restriction \( p|: p^{-1}(U) \to U \) is a Serre fibration. Then \( p: E \to B \) is itself a Serre fibration.

The proof of this theorem is relatively straightforward if we assume the following lemma. Recall the following notation from a previous lecture. Let \( F = \{ F_a | a \in A \} \) be a family of faces of the cube \( I^n \), and let

\[
J^n_{(F)} = (I^n \times \{0\}) \cup \left( \bigcup_a F_a \times I \right) \to I^n \times I
\]

be the inclusion.

**Lemma 6.12.** A map \( p: E \to B \) is a Serre fibration if and only if it has the RLP with respect to all maps of the form \( J^n_{(F)} \to I^n \times I \).

Note that the ‘if’-part is clear because the case \( F = \emptyset \) gives the definition of a Serre fibration. Earlier on, we have also used the case where \( F \) is the family of all the faces, when \( J^n_{(F)} \to I^{n+1} \) is homeomorphic to \( I^n \times \{0\} \to I^{n+1} \). The same is actually true for an arbitrary family \( F \), but one can also use an inductive argument to reduce the general case to the two cases where \( F = \emptyset \) or \( F \) consists of all faces. We will do this after the proof of Theorem 6.11.

**Proof of Theorem 6.11.** Let \( p: E \to B \) be as in the statement of the theorem, and consider a diagram of solid arrows of the form

\[
\begin{array}{ccc}
I^{n-1} \times \{0\} & \xrightarrow{g} & E \\
in \downarrow & & \downarrow \pi' \\
I^n & \xrightarrow{h} & B \\
\end{array}
\]

in which we wish to find a diagonal \( h \) as indicated. By assumption on \( p \) and compactness of \( I^n \), we can find a natural number \( k \) large enough so that for any sequence \((i_1, \ldots, i_n)\) of numbers \( 0 \leq i_1, \ldots, i_n \leq k-1 \), the small cube

\[
[i_1/k, (i_1 + 1)/k] \times \ldots \times [i_n/k, (i_n + 1)/k]
\]

is mapped by \( f \) into an open set \( U \subseteq B \) over which \( p \) is a Serre fibration (use the Lebesgue lemma!). Now order all these tuples \((i_1, \ldots, i_n)\) lexicographically, and list them as \( C_1, \ldots, C_{k^n} \). We will define a lift \( h \) by consecutively finding lifts \( h_r \) on \( C_1 \cup \ldots \cup C_r \subseteq I^n \) making the diagram

\[
\begin{array}{ccc}
I^{n-1} \times \{0\} & \xrightarrow{g} & E \\
in \downarrow & & \downarrow \pi' \\
C_1 \cup \ldots \cup C_r & \xrightarrow{h_r} & B \\
\end{array}
\]
commute. We can find $h_1$ because $(I^{n-1} \times \{0\}) \cap C_1$ is a small copy of $I^{n-1} \times \{0\} \to I^n$. And given $h_r$, we can extend it to $h_{r+1}$ by defining $h_{r+1}|_{C_{r+1}}$ as a lift in

$$
\begin{array}{c}
C_{r+1} \cap \left((I^{n-1} \times \{0\}) \cup (C_1 \cup \ldots \cup C_r)\right) \\
E
\end{array}
\xrightarrow{(g,h_r)}
\begin{array}{c}
\downarrow^p \\
B
\end{array}
\xrightarrow{h_{r+1}}
\begin{array}{c}
C_{r+1}
\end{array}
\xleftarrow{f}
\begin{array}{c}
I^n \times \{0\}
\end{array}
\xrightarrow{p}
\begin{array}{c}
B.
\end{array}
$$

Such a lift exists, because $C_{r+1} \cap \left((I^{n-1} \times \{0\}) \cup (C_1 \cup \ldots \cup C_r)\right) \to C_{r+1}$ is (essentially) a small copy of an inclusion $J^n_{(F)} \to I^n \times \{0\}$. (You should draw some pictures for yourself in the cases $n = 2, 3$ to see what is going on.)

Proof of Lemma 6.12. As we already said it only remains to establish the ‘only if’-direction which we already know in the cases of $F = \emptyset$ or the collection of all faces. We will reduce the intermediate cases to the case of all the faces by induction on $n$. For $n = 0$, $I^0$ has no faces so only the case $A = \emptyset$ applies and there is nothing to prove. For $n = 1$, there are four cases,

$$
A = \emptyset, \quad A = \{0\}, \quad A = \{1\}, \quad \text{and} \quad A = \{0, 1\},
$$

of which the first and the last have already been dealt with. For the intermediate case $A = \{0\}$, for example, consider a diagram of the form

$$
\begin{array}{c}
I^n \times \{0\} \cup \{0\} \times I \\
E
\end{array}
\xrightarrow{f}
\begin{array}{c}
I^n \times I \\
B
\end{array}
\xrightarrow{p}
\begin{array}{c}
I^2
\end{array}
\xrightarrow{f}
\begin{array}{c}
B
\end{array}
$$

where $p: E \to B$ is a Serre fibration. Now one can first find a lift for

$$
\begin{array}{c}
\{1\} \times \{0\} \\
E
\end{array}
\xrightarrow{g}
\begin{array}{c}
\{1\} \times I \\
B
\end{array}
$$

by the case $n = 0$. Then next fill the following diagram

$$
\begin{array}{c}
(I \times \{0\} \cup \{0\} \times I) \cup \{1\} \times I \\
E
\end{array}
\xrightarrow{(f,g)}
\begin{array}{c}
I^n \times I \\
B
\end{array}
\xrightarrow{p}
\begin{array}{c}
I^2
\end{array}
\xrightarrow{f}
\begin{array}{c}
B
\end{array}
$$

by the case where $F$ consists of all the faces (the fourth case).

The induction from $n$ to $n + 1$ proceeds in exactly the same way: suppose $F = \{F_a \mid a \in A\}$ is a family of faces of $I^{n+1}$ for which we wish to find a lift in a diagram of the form

$$
\begin{array}{c}
J^n_{(F)} \\
E
\end{array}
\xrightarrow{f}
\begin{array}{c}
I^{n+1} \times I \\
B
\end{array}
\xrightarrow{p}
\begin{array}{c}
I^{n+1}
\end{array}
$$
If $F$ does not consist of all the faces, we add a face $G$ and extend $f$ to $J^{n+1}_F \cup (G \times I) = J^{n+1}_{F \cup \{G\}}$ by lifting in

\[ G \times \{0\} \cup \bigcup_a (G \cap F_a) \times I \xrightarrow{\sim} E \]
\[ G \times I \xrightarrow{\sim} B \]

which is possible by the earlier case of the induction, because $\cup_a (G \cap F_a)$ is a family of faces of a cube of lower dimension. After having done this for all the faces not in $F$, we arrive at the case where $A$ is the set of all faces which was already settled. \qed
LECTURE 7: CW COMPLEXES AND BASIC CONSTRUCTIONS

In this section we will introduce CW complexes, which give us an important class of spaces which can be built inductively by gluing ‘cells’. Here we will study basic notions and examples, some facts concerning the point set topology of these spaces, and also give elementary constructions. In later lectures we will study homotopical properties of CW complexes. Moreover, we will see that this theory allows us to perform interesting constructions.

By the very definition, a CW complex is given by a space which admits a filtration such that each next filtration step is obtained from the previous one by attaching cells. Let us begin by introducing this process. Let \( e^n = \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\} \) be a copy of the (closed) \( n \)-ball. Its boundary \( \partial e^n = S^{n-1} \) is the \( (n-1) \)-sphere (for \( n = 0 \) we take \( \partial e^0 = \emptyset \)). If \( X \) is any space and \( \chi: \partial e^n \to X \), one can form a new space \( X \cup_\chi e^n \) as the pushout:

\[
\begin{array}{ccc}
\partial e^n & \xrightarrow{\chi} & X \\
i & & \downarrow \\
e^n & \xrightarrow{i} & X \cup_\chi e^n
\end{array}
\]

More explicitly, \( X \cup_\chi e^n \) is the space obtained from the disjoint union \( X \sqcup e^n \) by identifying each \( i(y) \in e^n \) with \( \chi(y) \in X \) for all \( y \in \partial e^n \), and equipping the resulting set with the quotient topology. The universal property of this quotient is as follows.

**Exercise 7.1.**

(i) The maps \( X \to X \cup_\chi e^n \) and \( e^n \to X \cup_\chi e^n \) are continuous and make the above square commutative. Moreover, the triple consisting of the space \( X \cup_\chi e^n \) and these two maps is initial with respect to this property. In other words, for all triples \( (W, g, h) \) consisting of a topological space \( W \) and continuous maps \( g: X \to W \) and \( h: e^n \to W \) such that the outer square in the following diagram commutes

\[
\begin{array}{ccc}
\partial e^n & \xrightarrow{\chi} & X \\
i & & \downarrow \\
e^n & \xrightarrow{i} & X \cup_\chi e^n \\
 & & \downarrow \\
 & & W
\end{array}
\]

then there is a unique dashed arrow \( X \cup_\chi e^n \to W \) such that the two triangles commute.

(ii) Define more generally the notion of a pushout for two arbitrary maps \( A \to X \) and \( A \to Y \) of spaces with a common domain. Show that the pushout exists and is unique up to a unique isomorphism in a way which is compatible with the structure maps.

(iii) Recall the notion of a pullback from a previous lecture and compare the two notions. These two notions are dual to each other. Compare also the actual constructions of pushouts and pullbacks in the category of spaces and see in which sense they are dual.
The notion of a pushout makes sense in every category but does not necessarily exist. To familiarize yourself with the concept, show that the categories \( \text{Set} \) and \( \text{Ab} \) have pushouts by giving an explicit construction.

We refer to the space \( X \cup e^n \) as being obtained from \( X \) by ‘attaching an \( n \)-cell’, and call \( \chi: \partial e^n \to X \) the \textit{attaching map}, and \( e^n \to X \cup e^n \) the \textit{characteristic map} of the ‘cell’ \( e^n \). Note that this characteristic map restricts to a homeomorphism of the interior of \( e^n \) to its image in \( X \cup e^n \), i.e., we have a relative homeomorphism \((e^n, \partial e^n) \to (X \cup e^n, X)\). The image of this homeomorphism is called the \textit{open cell}, and the image of \( e^n \to X \cup e^n \) the \textit{closed cell} of this attachment.

Usually one attaches more than one cell, and writes \( e_\sigma \) for the cell with ‘index \( \sigma \)’, sometimes leaving the dimension implicit. If \( \chi: \partial e_\sigma \to X \) is the attaching map, it is handy to freeze the index \( \sigma \), and write \( \chi_{\partial e_\sigma} \) for the attaching map, \( \chi_\sigma \) for the characteristic map, and refer to \( e_\sigma \) or its image as the cell (with index) \( \sigma \).

Thus if we obtain \( Y \) from \( X \) by attaching a set \( J_n \) of \( n \)-cells, then, by considering \( J_n \) as a discrete space, we have a pushout diagram of the following form:

\[
\begin{array}{ccc}
J_n \times \partial e^n & \rightarrow & X \\
\downarrow & & \downarrow \\
J_n \times e^n & \rightarrow & Y
\end{array}
\]

In particular, a subset of \( Y \) is open if and only if its preimages in \( X \) and each \( e^n \) are open, i.e., \( Y \) carries the quotient topology.

\textbf{Definition 7.2.} Let \( X \) be a topological space. A \textit{CW decomposition} of \( X \) is a sequence of subspaces

\[
X(0) \subseteq X(1) \subseteq X(2) \subseteq \ldots, \quad n \in \mathbb{N},
\]

such that the following three conditions are satisfied:

\begin{itemize}
  \item[(i)] The space \( X^{(0)} \) is discrete.
  \item[(ii)] The space \( X^{(n)} \) is obtained from \( X^{(n-1)} \) by attaching a (possibly) infinite number of \( n \)-cells \( \{e^n_\sigma\}_{\sigma \in J_n} \) via attaching maps \( \chi_\sigma: \partial e^n_\sigma \to X^{(n-1)} \).
  \item[(iii)] We have \( X = \bigcup X^{(n)} \) with the \textit{weak} topology (this means that a set \( U \subseteq X \) is open if and only if \( U \cap X^{(n)} \) is open in \( X^{(n)} \) for all \( n \geq 0 \)).
\end{itemize}

A CW decomposition is called \textit{finite} if there are only finitely many cells involved. A (finite) CW complex is a space \( X \) equipped with a (finite) CW decomposition. Given a CW decomposition of a space \( X \) then the subspace \( X^{(n)} \) is called the \( n \)-\textit{skeleton} of \( X \).

\textbf{Remark 7.3.}

\begin{itemize}
  \item[(i)] Note that by the very definition a CW complex is a space together with an additional structure given by the CW decomposition. Nevertheless, we will always only write \( X \) for a topological space endowed with a CW decomposition.
  \item[(ii)] Condition (iii) in Definition 7.2 is only needed for infinite complexes.
  \item[(iii)] From the definition of the weak topology it also follows that closed subsets of \( X \) can be detected by considering the intersections with all skeleta \( X^{(n)} \).
  \item[(vi)] The image of a characteristic maps \( \chi_\sigma: e_\sigma \to X \) is called a \textit{closed cell} in \( X \), and the image of \( \chi_\sigma: e^n_\sigma \to X \) an \textit{open cell}. These need not be open in \( X \)! Every point of \( X \) lies in a unique open cell.
(v) Each $X^{(n)}$ is a closed subspace of $X^{(n+1)}$, and hence of $X$. (The open $(n+1)$-cells are open in $X^{(n+1)}$ but not necessarily in $X$).

**Example 7.4.**

(i) The interval $I = [0, 1]$ has a CW decomposition with two 0-cells and one 1-cell by identifying the boundary of the unique 1-cell with the two 0-cells as expected.

(ii) The circle $S^1$ has a CW decomposition with one 0-cell and one 1-cell and no other cells. Of course, it also has a CW composition with two 0-cells and two 1-cells.

(iii) More generally, if one identifies the boundary $\partial e^n$ of the $n$-ball to a point, one obtains (a space homeomorphic to) the $n$-sphere. Thus the $n$-sphere has a CW decomposition with one 0-cell and one $n$-cell, and no other cells. One can also build up the $n$-sphere by starting with two points, then two half circles to form $S^1$, then two hemispheres to form $S^2$, and so on. Then $S^n$ has a CW decomposition with exactly 2 $i$-cells for $i = 0, \ldots, n$ (draw a picture for $n \leq 2$). If we take the coordinates $(x_0, \ldots, x_n)$ with $\sum x_i^2 = 1$ for $S^n$ as before, these two $i$-cells are

$$ e^i_+ = \{(x_0, \ldots, x_i, 0, \ldots, 0) \in S^n \mid x_i \geq 0\} $$

and

$$ e^i_- = \{(x_0, \ldots, x_i, 0, \ldots, 0) \in S^n \mid x_i \leq 0\}. $$

(iv) The real projective space $\mathbb{RP}^n$, the space of lines through the origin in $\mathbb{R}^{n+1}$, can be constructed as the quotient $S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts on the $n$-sphere by the antipodal map; in other words, by the quotient of $S^n$ obtained by identifying $x$ and $-x$. This identification maps the cell $e^i_+$ to $e^i_-$. Thus $\mathbb{RP}^n$ has a CW decomposition with exactly one $i$-cell for $i = 0, \ldots, n$. Recall from the previous lecture, that the Grassmannian varieties $G_{k,n}(\mathbb{R})$ parametrize $k$-planes in $\mathbb{R}^n$. Thus, we have $\mathbb{RP}^n \cong G_{1,n+1}(\mathbb{R})$.

(v) The complex projective space $\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. Such a line is determined by a point $(z_0, \ldots, z_n) \neq 0$ on the line, and for any scalar $\lambda \in \mathbb{C} \setminus \{0\}$ the tuple $(\lambda z_0, \ldots, \lambda z_n)$ determines the same line for which we write $[z_0, \ldots, z_n]$. The line can also be represented by a point $z = (z_0, \ldots, z_n)$ with $|z| = 1$, so that $z$ and $\lambda z$ represent the same line for all $\lambda \in S^1$. Thus $\mathbb{CP}^n = S^{2n+1}/S^1$ is a space of (real) dimension $2n$. There are inclusions

$$ * = \mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots $$

where $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ sends $[z_0, \ldots, z_n]$ to $[z_0, \ldots, z_{n-1}, 0]$. An arbitrary point in $\mathbb{CP}^n - \mathbb{CP}^{n-1}$ can be uniquely represented by $(z_0, \ldots, z_{n-1}, t)$ where $t > 0$ is the real number $\sqrt{1 - \sum z_i \overline{z}_i}$. This defines a map

$$ e^{2n} \to \mathbb{CP}^n : z = (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t] $$

with $t = \sqrt{1 - \|z\|^2}$. The boundary of $e^{2n}$ (where $t = 0$) is sent to $\mathbb{CP}^{n-1}$. In this way, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching one $2n$-cell. So $\mathbb{CP}^n$ has a CW structure with one cell in each even dimension $0, 2, \ldots, 2n$. Similarly to the previous example, we have an identification $\mathbb{CP}^n \cong G_{k,n+1}(\mathbb{C})$.

(vi) Every compact manifold is homotopy equivalent to a CW complex. (This is a theorem which we only include to indicate the generality of the notion.)

(vii) As we will see in a later lecture, every topological space is weakly homotopy equivalent to a CW complex.
Exercise 7.5.

(i) The torus $T$ can be obtained from the square by identifying opposite sides. Use an adapted CW decomposition of the square to also turn the torus into a CW complex.

(ii) Similarly we can obtain the Klein bottle from the unit square by identifying $(0, t) \sim (1, t)$ and $(s, 0) \sim (1 - s, 1)$. Show that there is a similar CW decomposition of the Klein bottle.

(iii) Can you come up with CW decompositions of the torus and the Klein bottle which have the same number of cells in each dimension? In particular this shows the obvious fact that the number of cells does not determine the space.

Lemma 7.6. Let $X$ be a CW complex and let $U$ be a subset of $X$. Then a subset $U \subset X$ is open if and only if $U \cap X^{(n)}$ is open for each $n$ if and only if $\chi_{\sigma}^{-1}(U) \subseteq e_\sigma^n$ is open for each cell $\sigma$ of $X$.

Proof. The equivalence of the first two statements holds true by definition of CW complexes. It is immediate that the second condition implies the third one. We want to prove the converse implication by induction so let us begin by observing that $U \cap X^{(0)}$ is open in $X^{(0)}$ since $X^{(0)}$ is discrete. For the inductive step, let us assume that $U \cap X^{(n-1)}$ is open in $X^{(n-1)}$ for some $n \geq 1$. Recall that we then have a pushout diagram of the following form:

$$\begin{array}{ccc}
J_n \times \partial e^n & \longrightarrow & X^{(n-1)} \\
\downarrow & & \downarrow \\
J_n \times e^n & \longrightarrow & X^{(n)}
\end{array}$$

By assumption $\chi_{\sigma}^{-1}(U) \subseteq e_\sigma^n$ is open for every $\sigma \in J_n$. But the above pushout diagram together with the induction assumption then tells us that also $U \cap X^{(n)}$ is open in $X^{(n)}$ concluding the proof. \qed

Thus, given a CW complex $X$ with $n$-cells parametrized by index sets $J_n$, then taking all the attaching maps together we obtain a map

$$(\chi_{\sigma})_{n, \sigma}: \bigsqcup_n J_n \times e^n \cong \bigsqcup_n \bigsqcup_{\sigma \in J_n} e_\sigma^n \longrightarrow X.$$ 

The above lemma shows that $X$ carries the quotient topology with respect to this map.

Corollary 7.7. Let $X$ be a CW complex, $Y$ a topological space, and $g: X \rightarrow Y$ a map of sets. Then the following are equivalent:

(i) The map $g: X \rightarrow Y$ is continuous.

(ii) The restriction $g|: X^{(n)} \rightarrow Y$ is continuous for all $n \geq 0$.

(iii) The map $g \circ \chi_{\sigma}: e_\sigma^n \rightarrow Y$ is continuous for each cell $e_\sigma^n$.

This corollary allows us to build continuous maps ‘cell by cell’. Thus, not only CW complexes can be built inductively by attaching cells but the same holds also true for maps defined on a CW complex. There is also a similar result for homotopies.

Exercise 7.8. Let $X$ be a CW complex, $Y$ a topological space, and $H: X \times I \rightarrow Y$ a map of sets. Then $H$ is continuous if and only if each composition

$$H \circ (\chi_{\sigma} \times id_I): e_\sigma^n \times I \longrightarrow X \times I \longrightarrow Y$$

is continuous for each cell $e_\sigma^n$ of $X$. 

Before turning to CW subcomplexes and an adapted class of morphisms, let us establish some more fundamental properties of CW complexes.

**Exercise 7.9.** A CW complex is normal. Thus show that disjoint closed subsets have disjoint open neighborhoods and that points are closed.

In studying the topology of CW complexes, one often uses the following fact.

**Proposition 7.10.** Any compact subset of a CW complex is contained in finitely many open cells.

This proposition in fact immediately follows from the following statement, by choosing a point in every open cell that intersects non-trivially the given compact subset.

**Lemma 7.11.** Let $X$ be a CW complex and $A \subseteq X$ a subspace. If $A$ has at most one point in each open cell then $A$ is closed in $X$ and the subspace topology on $A$ is discrete.

**Proof.** We check this by induction on $n$ and for each $A \cap X^{(n)}$ as a subspace of $X^{(n)}$. For $n = 0$ there is nothing to prove since $X^{(0)}$ is discrete. Suppose the statement has been proved for $A \cap X^{(n-1)} \subseteq X^{(n-1)}$. Write $A \cap X^{(n)} = B \sqcup C$ where $B = A \cap X^{(n-1)}$ and $C = A \cap (X^{(n)} - X^{(n-1)})$. Then $C$ is open in $A$ because the open $n$-cells are open in $X^{(n)}$, and for the same reason $C$ is discrete. The set $C$ is closed in $X^{(n)}$ because if $x \in C$ then $x$ lies in the same open cell as any point $c \in C$ close to $x$, hence $x = c$. So $C$ is closed and discrete in $X^{(n)}$. Also $B$ is closed and discrete in $X^{(n-1)}$ by induction hypothesis, hence in $X^{(n)}$ because $X^{(n-1)} \subseteq X^{(n)}$ is closed. Then $B \sqcup C$ has the same properties, which completes the induction step. \qed

**Remark 7.12.** This proposition allows us to explain the terminology ‘CW complex’. In the original definition given by J.H.C. Whitehead, the following two properties played a more essential role:

(C): The closure of every cell lies in a finite subcomplex (‘closure finite’).

(W): A subset is open if and only if it is open in the $n$-skeleton for all $n$ (‘weak topology’).

We now turn to an adapted class of morphisms between CW complexes.

**Definition 7.13.** A map $f : X \to Y$ between CW complexes is cellular if it satisfies $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n$. It is immediate that we have a category CW of CW complexes and cellular maps.

Thus, such a cellular map induces commutative diagrams of the form:

$$
\begin{array}{ccc}
X^{(n)} & \xrightarrow{f} & Y^{(n)} \\
\downarrow{i} & & \downarrow{i} \\
X & \xrightarrow{f} & Y
\end{array}
$$

Let us give some examples of cellular maps. We will see in a later lecture that this notion is rather generic.

**Example 7.14.**

(i) The vector space $\mathbb{R}^n$ maps injectively to $\mathbb{R}^{n+1}$ by adding a zero as the last coordinate, i.e., we have a map

$$i_n : \mathbb{R}^n \to \mathbb{R}^{n+1} : (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 0).$$
These maps restrict to maps of spheres as follows
\[ j_n = i_{n+1} : S^n \rightarrow S^{n+1} \]
and these maps are cellular with respect to the CW decompositions on the spheres with precisely two cells in each dimension lower or equal to the dimension of the respective sphere (but not with respect to the other one).

(ii) Since the inclusions \( i_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \) are compatible with the actions by \( \mathbb{R} \times \), we obtained induced maps \( j'_n : \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^n \) which are easily seen to be cellular with respect to the CW decomposition of Example 7.4. The maps are also obtained from the maps \( j_n \) of the last example by passing to the quotient of the \( \mathbb{Z}/2\mathbb{Z} \)-action and these quotient maps are also cellular. Thus, we have a diagram of cellular maps:

\[
\begin{array}{cccccc}
S^0 & \xrightarrow{j_0} & S^1 & \xrightarrow{j_1} & S^2 & \xrightarrow{j_2} \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\mathbb{R}P^0 & \xrightarrow{j'_0} & \mathbb{R}P^1 & \xrightarrow{j'_1} & \mathbb{R}P^2 & \xrightarrow{j'_2} \cdots \\
\end{array}
\]

Similarly, in the case of complex numbers, we have cellular maps:

\[
\mathbb{C}P^0 \rightarrow \mathbb{C}P^1 \rightarrow \mathbb{C}P^2 \rightarrow \cdots
\]

(iii) In Example 7.4 we introduced two CW decompositions on the \( n \)-sphere. Let us write \( S^n \) for the one with two cells in each dimension \( d \leq n \) while we write \( \widehat{S}^n \) for the one with precisely one 0-cell and one \( n \)-cell. Then the identity map \( \text{id} : \widehat{S}^n \rightarrow S^n \) is cellular, while this is not the case for \( \text{id} : S^n \rightarrow \widehat{S}^n \) if \( n \geq 2 \).

We now turn to subcomplexes of CW complexes.

**Proposition 7.15.** Let \( X \) be a CW complex and let \( Y \subset X \) be a closed subspace such that the intersection \( Y \cap (X^{(n)} - X^{(n-1)}) \) is the union of open \( n \)-cells. The filtration

\[ Y^{(0)} \subset Y^{(1)} \subset \cdots \subset Y \]

given by \( Y^{(n)} = Y \cap X^{(n)} \) then defines a CW decomposition on \( Y \). Moreover, the inclusion \( Y \rightarrow X \) is then a cellular map.

This proposition allows us to introduce pointed CW complexes and pairs of CW complexes.

**Definition 7.16.** In the notation of the above proposition, we refer to \( Y \) as a CW subcomplex of \( X \) and to \( (X,Y) \) as a CW pair. A pointed CW complex \((X,x_0)\) is a CW complex \( X \) together with a chosen base point \( x_0 \in X^{(0)} \).

In the obvious way, this gives us the category of pointed CW complexes and CW pairs whose definitions are left to the reader.

**Example 7.17.**

(i) For an arbitrary CW complex \( X \), we have CW pairs \((X,X^{(n)})\) for all \( n \) and similarly \((X^{(n)},X^{(m)})\) for \( n \geq m \).

(ii) We have CW pairs \((S^n,S^m)\), \((
\mathbb{R}P^n,\mathbb{R}P^m)\) and similarly in the complex case for \( n \geq m \). If we endow the unions

\[ S^\infty = \bigcup_n S^n, \quad \mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n, \quad \mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n \]
with the weak topology then each of the three spaces carries canonically a CW structure. Moreover, we have CW pairs \((S^n, S^n)\), \((\mathbb{R}P^n, \mathbb{R}P^n)\), and \((\mathbb{C}P^n, \mathbb{C}P^n)\) for all \(n\).

**Exercise 7.18.** Let \((X,Y)\) be a CW pair. Then the quotient space \(X/Y\) can be turned in a CW complex such that the quotient map \(X \to X/Y\) is cellular.

We will now establish a few more closure properties of CW complexes. Let us begin with a more difficult one, namely the product. Recall that we observed that each CW complex is obtained from a disjoint union of cells by passing to a quotient space. Namely, for a CW complex \(X\) we have a quotient map:

\[
\bigsqcup_n J_n \times e^n \to X
\]

Given two CW complexes \(X\) and \(Y\) one might now try to take two such presentations

\[
\bigsqcup_n J_n(X) \times e^n \to X \quad \text{and} \quad \bigsqcup_m J_m(Y) \times e^m \to Y
\]

and use homeomorphisms \(e^n \times e^m \cong e^{n+m}\) to obtain a map

\[
\bigsqcup_k J_k(X \times Y) \times e^k \to X \times Y
\]

where \(J_k(X \times Y) = \bigsqcup_{n+m=k} J_n(X) \times J_m(Y)\). However, this map is, in general, not a quotient map. More conceptually, the problem is that the formation of products and quotients in the category of spaces are not compatible in general. Nevertheless, under certain ‘finiteness conditions’ one can obtain a positive result. We will give a proof of this result in a later lecture.

**Proposition 7.19.** Let \(X, K\) be CW complexes such that \(K\) is finite. Then the product \(X \times K\) is again a CW complex with the above CW decomposition.

**Proof.** Will be given in a later lecture. \(\square\)

Using the last proposition we can establish many more closure properties for the class of CW complexes.

**Corollary 7.20.**

(i) The coproduct of two CW complexes is again a CW complex such that the inclusions of the respective summands are cellular.

(ii) Given a CW complex \(X\) then the cylinder \(X \times I\) is again a CW complex. For each \(n\)-cell \(e^n_\sigma\) of \(X\) we obtain three cells for \(X \times I\), namely two \(n\)-cells \(e^n_\sigma \times \{0\}, e^n_\sigma \times \{1\}\), and an \((n+1)\)-cell \(e^n_\sigma \times e^1\). Moreover, the cylinder comes with cellular maps \(i_0, i_1 : X \to X \times I\) and \(p : X \times I \to X\).

(iii) Given a CW complex \(X\), then the unreduced suspension \(SX = (X \times I)/(X \times \partial I)\) is again a CW complex.

(iv) We have similar variants for the context of pointed CW complexes. The wedge product of pointed CW complexes is again a CW complex. Similarly, the reduced cylinder of a pointed CW complex is again a pointed CW complex. More generally, if \(K\) is a finite pointed CW complex and if \(X\) is a pointed CW complex, then so is the smash product \(X \wedge K\). In particular, the (reduced) suspension of a pointed CW complex is again a pointed CW complex.

**Proof.** The first statement is immediate while the other ones follow immediately from Exercise 7.18, Proposition 7.19, and Example 7.4. \(\square\)
In the definition of a CW complex $X$, the first condition we imposed was that $X^{(0)}$ is to be a discrete space and then that the higher skeleta are obtained from the lower ones by attaching $n$-cells for $n \geq 1$. We can also think of $X^{(0)}$ as being obtained from the empty space by attaching 0-cells; in fact, using the convention that $\partial e^0 = \emptyset$ we have a pushout:

\[
\begin{array}{ccc}
X^{(0)} \times \partial e^0 &=& \bigsqcup_{\sigma \in X_0} \partial e^0_{\sigma} \\
\downarrow & \cong & \downarrow \\
X^{(0)} \times e^0 &=& \bigsqcup_{\sigma \in J_0} e^0_{\sigma} \cong X^{(0)}
\end{array}
\]

This observation is more than only a rather picky remark since it motivates the following generalization of the notion of CW complex.

**Definition 7.21.** Let $(X, A)$ be a pair of spaces. Then $X$ is a **CW complex relative to $A$**, if there is a filtration of $X$,

\[
A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,
\]

such that the following two properties are satisfied:

(i) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching $n$-cells for $n \geq 0$.

(ii) The space $X$ is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology.

In this situation, the pair $(X, A)$ is called a **relative CW complex**.

**Example 7.22.**

(i) Let $X$ be a CW complex and $x_0 \in X_0$. Then we have a relative CW complex $(X, x_0)$.

(ii) More generally, every CW pair is a relative CW complex.

One point of the notion of a relative CW complex $(X, A)$ is that the associated inclusion map $A \to X$ is not an arbitrary map but has nice properties. In a way, these properties are dual to the ones of fibrations. We will come back to this in the next lecture where we will, in particular, talk about *cofibrations*. 
In this section we introduce the class of cofibration which can be thought of as nice inclusions. There are inclusions of subspaces which are ‘homotopically badly behaved’ and these will be excluded by considering cofibrations only. To be a bit more specific, let us mention the following two phenomenona which we would like to exclude. First, there are examples of contractible subspaces \( A \subseteq X \) which have the property that the quotient map \( X \to X/A \) is not a homotopy equivalence. Moreover, whenever we have a pair of spaces \((X,A)\), we might be interested in extension problems of the form:

\[
\begin{array}{c}
\begin{array}{ccc}
A & \longrightarrow & W \\
\downarrow & & \downarrow \\
X & \longrightarrow & W
\end{array}
\end{array}
\]

Thus, we are looking for maps \( h \) as indicated by the dashed arrow such that \( h \circ i = f \). In general, it is not true that this problem ‘lives in homotopy theory’. There are examples of homotopic maps \( f \simeq g \) such that the extension problem can be solved for \( f \) but not for \( g \). By design, the notion of a cofibration excludes this phenomenon.

**Definition 8.1.**

(i) Let \( i : A \to X \) be a map of spaces. The map \( i \) has the **homotopy extension property** with respect to a space \( W \) if for each homotopy \( H : A \times [0,1] \to W \) and each map \( f : X \times \{0\} \to W \) such that \( f(i(a),0) = H(a,0), \ a \in A \), there is map \( K : X \times [0,1] \to W \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X \times \{0\} \cup A \times \{0\} & \longrightarrow & X \times [0,1] \\
\downarrow & & \downarrow \\
A \times [0,1] & \longrightarrow & W
\end{array}
\]

(ii) A map \( i : A \to X \) is a **cofibration** if it has the homotopy extension property with respect to all spaces \( W \).

The terminology ‘homotopy extension property’ is of course motivated by the case of an inclusion of a subspace. And, in fact, it turns out that an arbitrary cofibration \( i : A \to X \) is always injective.

The space showing up in the definition of a cofibration is a pushout. In the case of the inclusion of a closed subspace there is the following simplification.

**Lemma 8.2.** Let \( i : A \to X \) be the inclusion of a closed subspace. Then we have a homeomorphism \( \phi : X \times \{0\} \cup_{A \times \{0\}} A \times [0,1] \to X \times \{0\} \cup A \times [0,1] \) which is compatible with the maps
to $X \times [0, 1]$, that is, which makes the following diagram commute:

$$
\begin{array}{c}
X \times \{0\} \cup_{A \times \{0\}} A \times [0, 1] \xrightarrow{\phi} X \times \{0\} \cup A \times [0, 1] \\
\downarrow \\
X \times [0, 1].
\end{array}
$$

**Proof.** The universal property of the pushout allows us to construct a map as follows:

$$
\begin{array}{c}
X \times \{0\} \\
\downarrow \\
A \times \{0\}
\end{array} \quad \begin{array}{c}
A \times \{0\} \\
\downarrow \\
A \times [0, 1]
\end{array} \quad \begin{array}{c}
A \times [0, 1] \\
\downarrow \\
X \times \{0\} \cup A \times [0, 1]
\end{array} \quad \begin{array}{c}
X \times \{0\} \cup A \times [0, 1] \\
\downarrow \\
X \times \{0\}
\end{array}
$$

Here the square on the left is the pushout diagram while the two bent arrows are just the inclusions. The universal property gives us the continuous map $\phi$ which is easily seen to be a bijection. Now, it suffices to observe that $\phi$ is also a closed map so that we actually have a homeomorphism. By definition of the quotient topology it is enough to check that the two inclusions (the bent arrows) are closed maps which is always the case for the upper one. Since $A$ is assumed to be a closed subspace of $X$ also the lower inclusion is a closed map. Thus, we have shown that $\phi$ is a homeomorphism and it is immediate that it is compatible with the two maps to $X \times [0, 1]$. \qed

Using this lemma we can now easily establish the following convenient criterion which allows us to identify certain maps as being cofibrations.

**Proposition 8.3.** Let $A \subseteq X$ be a closed subspace. Then the following are equivalent:

(i) The inclusion $i: A \rightarrow X$ is a cofibration.

(ii) All extension problems of the form

$$
\begin{array}{c}
X \times \{0\} \cup A \times [0, 1] \xrightarrow{f} W \\
\downarrow \\
X \times [0, 1]
\end{array}
$$

admit a solution as indicated by the dashed arrow.

(iii) The map $j: X \times \{0\} \cup A \times [0, 1] \rightarrow X \times [0, 1]$ admits a retraction.

**Proof.** The equivalence of the first two statements follows immediately from the previous lemma. It is also easy to see that (ii) implies (iii) since it suffices to consider the lifting problem given by the identity:

$$
\begin{array}{c}
X \times \{0\} \cup_{A \times \{0\}} A \times [0, 1] \xrightarrow{\text{id}} X \times \{0\} \cup A \times [0, 1] \\
\downarrow \\
X \times [0, 1]
\end{array}
$$

\qed
Finally, if \( j \) admits a retraction \( r \) then any extension problem as in (ii) admits a solution given by \( f \circ r \).

Recall from a previous lecture that the inclusion

\[
j_n: J^n = I^n \times \{0\} \cup \partial I^n \times I \to I^n \times I
\]

admits a retraction. It is immediate that our preferred homeomorphism \( D^n \cong I^n \) (which restricts to a homeomorphism \( S^{n-1} \cong \partial I^n \)) shows that also the inclusion

\[
D^n \times \{0\} \cup S^{n-1} \times I \to D^n \times I
\]

admits a retraction. Thus, an application of the previous proposition gives us the following example.

**Example 8.4.** The inclusion \( S^{n-1} \to D^n \) is a cofibration.

Our next aim is to show that if \((X,A)\) is a relative CW complex, then the inclusion \( A \to X \) is a cofibration. The above example gives us one of the basic building blocks. Since relative CW complexes are built inductively using certain constructions it is convenient to first establish some ‘closure properties’ of the class of cofibration.

We begin by observing that there is the following reformulation. Recall that associated to an arbitrary space \( W \) there is the path space \( W^I \) (endowed with the compact-open topology) which comes with natural evaluation maps \( W^I \to W \).

**Lemma 8.5.** A map \( i: A \to X \) is a cofibration if and only if for all spaces \( W \) and commutative diagrams of the form

\[
\begin{array}{ccc}
A & \to & W^I \\
\downarrow i & & \downarrow e_0 \\
X & \to & W
\end{array}
\]

there is a diagonal filler as indicated, i.e., such a map making both triangles commutative.

**Proof.** This follows immediately from the fact that we have a natural bijection between the set of maps \( X \times I \to W \) and maps \( X \to W^I \) (see Proposition 2.1, Lecture 2). The naturality of these bijections is essential to conclude the proof and the details are left as an exercise.

This lemma allows us to establish the following closure properties of cofibrations.

**Proposition 8.6.**

(i) Homeomorphisms are cofibrations. Similarly, if we have maps \( i: A \to X \), \( i': A' \to X' \), and homeomorphisms \( A \cong A' \), \( X \cong X' \) such that

\[
\begin{array}{ccc}
A & \cong & A' \\
i & & i' \\
X & \cong & X'
\end{array}
\]

commutes, then \( i \) is a cofibration if and only if \( i' \) is one.

(ii) Cofibrations are closed under composition, that is, if \( i: A \to X \) and \( j: X \to Y \) is a cofibration then so is \( j \circ i: A \to Y \).

(iii) Cofibrations are closed under coproducts, that is, if we have a family \( i_j: A_j \to X_j \), \( j \in J \), of cofibrations then also the map \( \bigsqcup i_j: \bigsqcup A_j \to \bigsqcup X_j \) is a cofibration.
(iv) Cofibrations are stable under pushouts, that is, if we have a pushout diagram
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^i & & \downarrow^j \\
X & \rightarrow & Y
\end{array}
\]
such that \(i\) is a cofibration, then also \(j\) is a cofibration.

(v) For two spaces \(X\) and \(Y\), the inclusion \(X \rightarrow X \sqcup Y\) is a cofibration. In particular, taking \(X\) to be the empty space, the map \(\emptyset \rightarrow Y\) is cofibration for every space \(Y\).

Proof. All of these facts follow more or less directly from the definition or the above lemma. We will give a proof of the stability under pushouts and leave the remaining ones as exercises. So, in the notation of (iv) let us consider the situation:
\[
\begin{array}{ccc}
A & \overset{f}{\rightarrow} & B & \overset{g}{\rightarrow} & W^l \\
\downarrow^i & & \downarrow^j & & \downarrow^e_0 \\
X & \overset{k}{\rightarrow} & Y & \overset{l}{\rightarrow} & W
\end{array}
\]
Thus, we are given the solid arrow diagram and we try to find a diagonal filler \(d\) as indicated. Using the fact that \(i\) is a cofibration we can find a solution \(e\) to the following problem on the left:
\[
\begin{array}{ccc}
A & \overset{gf}{\rightarrow} & W^l \\
\downarrow^i & & \downarrow^e_0 \\
X & \overset{kl}{\rightarrow} & W
\end{array}
\]
The commutativity of the upper triangle tells us that we have maps \(e: X \rightarrow W^l\) and \(g: B \rightarrow W^l\) such that the above square on the right commutes. The universal property of the pushout implies that there is a unique map \(d: Y \rightarrow W^l\) such that \(d \circ j = g\) and \(d \circ k = e\). We leave it to the reader to check that this map \(d\) does the job (which follows by using once more the universal property of pushouts).

Remark 8.7.
(i) Recall from the section on fibrations, that we introduced two such classes, namely the Hurewicz fibrations and the Serre fibrations. The cofibrations introduced in this section are often also referred to as Hurewicz cofibrations. Since we will not consider 'Serre cofibrations' in this course, we instead decided to simplify the terminology and drop the name Hurewicz.
(ii) For readers knowing about model categories we want to include this warning. The class of cofibrations introduced in this section is not the class of cofibrations in the Hurewicz model structure on the category of all spaces. Instead, the cofibrations in that model structure are given by the closed Hurewicz cofibrations. Nevertheless, it turns out that every object in the Hurewicz model structure is cofibrant, i.e., the unique map from the empty space to the given one in a closed Hurewicz cofibration. With this respect, this model category thus behaves vastly different than the Serre model structure on spaces. (It is even true that every object in the Hurewicz model structure is both cofibrant and fibrant.)

With these closure properties of cofibrations we can now deduce the following important result.
Theorem 8.8. Let \((X, A)\) be a relative CW complex. Then the inclusion \(i : A \rightarrow X\) is a cofibration.

Proof. We thus have to show that every problem of the form

\[
\begin{array}{c}
A \\ i \\
\downarrow \\
X
\end{array} \quad \begin{array}{c}
f \\
\downarrow \sigma_0 \\
W' \quad I
\end{array} \quad \begin{array}{c}
\downarrow \\
\epsilon_0
\end{array}
\]

admits a solution. By definition of a relative CW complex we have a filtration of \(X\),

\[A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,\]

such that the following two properties are satisfied:

(i) The space \(X^{(n)}\) is obtained from \(X^{(n-1)}\) by attaching \(n\)-cells for \(n \geq 0\).

(ii) The space \(X\) is the union \(\bigcup_{n \geq -1} X^{(n)}\) endowed with the weak topology and so comes, in particular, with continuous inclusions \(i_n : X^{(n)} \rightarrow X\).

Thus, for every \(n\) we have a pushout diagram

\[
\begin{array}{c}
J_n \times \partial e^n = \bigsqcup_{\sigma \in J_n} \partial e^n \\
\downarrow \\
J_n \times e^n = \bigsqcup_{\sigma \in J_n} e^n \\
\downarrow \\
X^{(n)},
\end{array}
\]

We know already that \(\partial e^n \rightarrow e^n\) is a cofibration (Example 8.4). Since cofibrations are stable under coproducts and pushouts we conclude that also the maps \(i_{n,n-1} : X^{(n-1)} \rightarrow X^{(n)}\) are cofibrations. But this means that we can inductively find solutions to the following problems

\[
\begin{array}{c}
A \quad \begin{array}{c} f \\
\downarrow i_0 \rightarrow W \quad I \quad \downarrow \sigma_0 \\
k_0 \rightarrow W
\end{array} \\
X^{(0)} \\
\begin{array}{c} d_0 \\
\downarrow d_1 \rightarrow W, \\
k_{i_0} \rightarrow W
\end{array}
\end{array} \quad \begin{array}{c}
X^{(0)} \quad \begin{array}{c} d_0 \\
\downarrow \sigma_0 \\
k_{i_0} \rightarrow W, \\
\begin{array}{c} d_0 \\
\downarrow d_1 \\
\downarrow d_2 \rightarrow W
\end{array}
\end{array}
\end{array}
\]

where in each step we use that the map on the left is a cofibration. Note that we use the inductively constructed solution \(d_n : X^{(n)} \rightarrow W^I\) as an input for the problem in the next dimension. Using the weak topology on \(X\), there is a unique map \(d : X \rightarrow W^I\) such that \(d \circ i_n = d_n : X^{(n)} \rightarrow W^I\), and hence, in particular, \(d \circ i = f : A \rightarrow W^I\). Thus the upper triangle in the initial problem commutes. We leave it to the reader to check that this map is a solution for the initial problem, that is, that also the lower triangle commutes. For that purpose, you will again have to use that \(X\) is endowed with the weak topology. □

This theorem tells us that plenty of cofibrations show up in nature. In fact, we now want to show that every map of spaces can be factored in a cofibration followed by a homotopy equivalence. This factorization uses the mapping cylinder construction which is obtained as follows. Let \(f : X \rightarrow Y\)
be a map of spaces. Then the mapping cylinder $M_f$ of $f$ is defined by the following pushout:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{i_2} & & \downarrow{j} \\
X \times I & \xrightarrow{k} & M_f
\end{array}
$$

(Note that the mapping cylinder construction was already used in the definition of a cofibration.) Thus, $M_f$ is obtained by gluing a cylinder $X \times I$ on $Y$ by identifying points $(x, 1) \sim f(x)$. The constant homotopy $f \circ p: X \times I \to X \to Y$ and the identity id: $Y \to Y$ are two maps with the same target and which satisfy $id \circ f = (f \circ p) \circ i_1: X \to Y$. Thus, by the universal property of the pushout we obtain a map $r: M_f \to Y$ which just ‘collapses the cylinder’. By design, this map $r$ satisfies the equations

$$
r \circ j = id: Y \to Y \quad \text{and} \quad r \circ k = f \circ p.
$$

The first equation of course tells us that $r$ is a retraction. We also have a map from $X$ to the mapping cylinder, given by the ‘inclusion as the top of the cylinder’, namely the map

$$
i = k \circ i_0: X \xrightarrow{i_0} X \times I \xrightarrow{k} M_f.
$$

The following proposition justifies our intuition that this map $i$ is a nice inclusion. Before we state it, let us recall the following definition.

**Definition 8.9.** Let $(Z, C)$ be a pair of spaces with inclusion $\iota: C \to Z$. The map $\iota$ is the inclusion of a **strong deformation retract** if it is the inclusion of a retract, that is, such that $r \circ \iota = id_C: C \to C$ together with a homotopy $H: \iota \circ r \simeq id_Z$ relative to $C$. The map $r: Z \to C$ is then called a **strong deformation retraction**.

**Proposition 8.10.** Let $f: X \to Y$ be a map of spaces as above. Then the map $f$ factors as

$$
f = r \circ i: X \to M_f \to Y.
$$

Moreover, $r: M_f \to Y$ is a strong deformation retraction and $i: X \to M_f$ a cofibration.

**Proof.** To check that we have such a factorization it suffices to make the following calculation:

$$
r \circ i = r \circ k \circ i_0 = f \circ p \circ i_0 = f.
$$

We know that $j: Y \to M_f$ is the inclusion of a retract with retraction given by $r: M_f \to Y$. Thus, to conclude that we have a deformation retraction it suffices to show that the map $j \circ r: M_f \to M_f$ is homotopic to the identity (relative to $Y$). The idea is of course to ‘linearly collapse’ the cylinder and keep the rest fixed. In formulas, consider the map $H: M_f \times I \to M_f$ by setting

$$
H([y, t]) = [y] \quad \text{and} \quad H([x, s], t) = [x, ts + (1 - t)].
$$

We leave it to the reader to check that this map is well-defined and continuous. Then it is immediate from the formula that $H(-, 0) = j \circ r$ and $H(-, 1) = id$. By construction, the homotopy is constant on $Y$ so that we know that $r$ is a strong deformation retraction.
It remains to show that \( i : X \to M_f \) is a cofibration. With our preparation, this is now easily established. In fact, note that there is a diagram of the following form

\[
\begin{array}{ccc}
X & \xrightarrow{i_1} & X \\
\downarrow & & \downarrow \\
X \sqcup X & \xrightarrow{id \cup f} & X \sqcup Y \\
\downarrow & & \downarrow \\
X \times I & \xrightarrow{(i,j)} & M_f
\end{array}
\]

in which the maps \( i \) are the inclusions of the first summands. It is easy to check that the composition in the right column is our map \( i : X \to M_f \) so that by Proposition 8.6 it is enough to check that both maps in right column are cofibrations. That same proposition already takes care of the first map. To show that also the second map is a cofibration we can again apply Proposition 8.6 to deduce that it is enough to show that \( X \times \partial I = X \sqcup X \to X \times I \) is a cofibration. Since this map is the inclusion of a closed subspace we can apply Proposition 8.3 to conclude that we only have to show that the map

\[
X \times \partial I \times \{0\} \cup X \times I \times \{0\} \to X \times I \times I
\]

admits a retraction. But this is immediate since we know that \( \partial I \times I \cup I \times \{0\} = J^1 \to I^2 \) admits a retraction. □

Now, a strong deformation retraction is, in particular, a homotopy equivalence so that we have managed factoring an arbitrary map into a cofibration followed by a homotopy equivalence. The mapping cylinder construction is helpful for other purposes as well. For example, it can be used to establish the following result.

**Exercise 8.11.** Let \( i : A \to X \) be a cofibration. Then \( i \) is injective.

Next, we will show that the pushout of a homotopy equivalence along a cofibration is again a homotopy equivalence.

**Proposition 8.12.** Let \( i : A \to X \) be a cofibration and let \( f, g : A \rightrightarrows Y \) be two maps. If \( f \) and \( g \) are homotopy, then the two pushouts \( X \cup_f Y \) and \( X \cup_g Y \) (of \( i \) and \( f \), and of \( i \) and \( g \), respectively) are homotopy equivalent spaces. And if the given homotopy between \( f \) and \( g \) respects the base point, so does the resulting homotopy equivalence.

**Proof.** Let us write

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow \varphi \\
X & \xrightarrow{\alpha} & W_f \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
A & \xrightarrow{g} & Y \\
\downarrow i & & \downarrow \psi \\
X & \xrightarrow{\beta} & W_g \\
\end{array}
\]

for the two pushouts (so \( W_f = X \cup_f Y \) and \( W_g = X \cup_g Y \)), and \( E : A \times I \to Y \) for the given homotopy, from \( f = E_0 \) to \( g = E_1 \). We are going to construct maps \( \lambda : W_f \to W_g \) and \( \mu : W_g \to W_f \), and show
that they are inverse up to homotopy. To get $\lambda$, we use the pushout property in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow{i} & & \downarrow{\varphi} \\
X & \xrightarrow{\alpha} & W_f \\
\downarrow{\beta} & & \downarrow{\psi} \\
W_g & \xrightarrow{\mu} & W_f
\end{array}
$$

for a suitable $\beta'$. We cannot use $\beta$, because we may not have $\beta \circ i = \psi \circ f$. However, $\beta \circ i = \psi \circ g \simeq \psi \circ f$, and by the homotopy extension property, we will be able to find a map $\beta' \simeq \beta$, with $\beta' \circ i = \psi \circ f$. Indeed, to this end consider the diagram

$$
(A \times X) \cup (X \times \{0\}) \xrightarrow{(\psi \circ E^{-1}) \cup \beta} W_g
$$

where $E^{-1}$ is the homotopy $E$ running in the opposite direction, $E^{-1}(a, t) = E(a, 1 - t)$, from $g$ to $f$, so that $(\psi \circ E^{-1})_0 = \psi \circ E_1 = \psi \circ g = \beta \circ i$. This shows that the map $(\psi \circ E^{-1}) \cup \beta$ on top of the diagram is well defined. The homotopy extension property provides a homotopy $H : X \times I \to W_g$ with $H|_{A \times I} = \psi \circ E^{-1}$ and $H_0 = \beta$, and we define $\beta' = H_1$.

Similarly, we find a map $\mu$ in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g} & Y \\
\downarrow{i} & & \downarrow{\varphi} \\
X & \xrightarrow{\beta} & W_g \\
\downarrow{\alpha'} & & \downarrow{\mu} \\
W_f & \xrightarrow{\lambda} & W_f
\end{array}
$$

where the map $\alpha'$ satisfies that $\alpha' \circ i = \varphi \circ g$ as in the diagram, and $\alpha'$ is found by applying the homotopy extension property in

$$
(A \times X) \cup (X \times \{0\}) \xrightarrow{(\varphi \circ E) \cup \alpha} W_f
$$

and then defining $\alpha' = K_1$.

We now claim that $\mu \circ \lambda \simeq \text{id}_{W_f}$ and $\lambda \circ \mu \simeq \text{id}_{W_g}$. These statements are, of course, proved in the same way, so let us prove the first. To find a homotopy $L : \mu \circ \lambda \simeq \text{id}_{W_f}$, it is natural to try
and use the pushout property, as in

\[
\begin{array}{c}
A \xrightarrow{f} Y \\
i \\
X \xrightarrow{\alpha} W_f \\
M \\
\end{array}
\]

and get \( L \) from two homotopies \( \mu \circ \lambda \circ \varphi \simeq \varphi \) and \( \mu \circ \lambda \circ \alpha \simeq \alpha \), which agree on \( A \times I \). Now \( \mu \circ \lambda \circ \varphi = \varphi \), so for \( C \) we can take the constant homotopy from \( \varphi \) to itself. It remains to find a homotopy \( M \) between \( \mu \circ \lambda \circ \alpha = \mu \circ \beta' \) and \( \alpha \), such that \( M|_{A \times I} \) is the constant homotopy \( A \times I \to A \xrightarrow{f} Y \xrightarrow{\varphi} W_f \).

Now, we do have a homotopy between \( \mu \circ \beta' \) and \( \alpha \), namely

\[
\alpha = K_0 \xrightarrow{K} K_1 = \alpha' = \mu \circ \beta = \mu \circ H_0 \xrightarrow{\mu \circ H} \mu \circ H_1 = \mu \beta',
\]

but when restricted to \( A \times I \) this is

\[
\varphi(E \ast E^{-1}) \circ i \xrightarrow{\varphi \circ E} \alpha' \circ i = \varphi \circ E_1 = \varphi \circ g \xrightarrow{\varphi \circ E^{-1}} \varphi \circ f = \alpha \circ i.
\]

This homotopy, first \( \varphi \circ E \) and then \( \varphi \circ E \) in the opposite direction, is obviously homotopic to the constant homotopy from \( \alpha \circ i \) to itself. Write \( N \) for this homotopy. So \( N: A \times I \times I \to W_f \) satisfies the following identities:

\[
\begin{align*}
N(a, s, 0) &= \varphi(E \ast E^{-1})(a, s), \\
N(a, s, 1) &= \alpha \circ i(a), \\
N(a, 0, t) &= \alpha \circ i(a) = N(a, 1, t).
\end{align*}
\]

Now apply the homotopy extension property again as in the diagram

\[
(A \times I \times I) \cup (X \times U) \xrightarrow{N \cup \chi} W_f
\]

(cf. Exercise 8.13 below) to find an \( M \) a required in diagram (1) above: here \( U \) is the \( U \)-shape \( \{0, 1\} \times I \cup I \times \{0\} \to I \times I \), and the map \( \chi: X \times U \to W_f \) is defined as follows:

\[
\begin{align*}
\alpha \text{ on } X \times \{0\} \times I, \\
\mu \circ \beta' \text{ on } X \times \{1\} \times I, \\
K \ast (\mu \circ H) \text{ on } X \times I \times \{0\}.
\end{align*}
\]

Now \( M = \overline{M}_1 \) is a homotopy as required in (1), and the pushout property of (1) will give the homotopy \( L \) form the identity to \( \mu \circ \lambda: W_f \to W_f \).

We leave the proof of the statement concerning the basepoints to the reader. \( \square \)
Exercise 8.13. Prove that if \( i : A \rightarrow X \) is a cofibration, then it has the homotopy extension property in diagrams of the form
\[
\begin{array}{c}
A \times I \times I \cup X \times U \\
\downarrow \\
X \times I \times I.
\end{array}
\]

Remark 8.14. Consider again the two pushout squares in the first sentence of the proof above which define \( W_f = X \cup_f Y \) and \( W_g = X \cup_g Y \). The homotopy equivalences \( \lambda : W_f \xrightarrow{\simeq} W_g \) and \( \mu : W_g \xrightarrow{\simeq} W_f \) are compatible up to homotopy with these squares, in the sense that
\[
\lambda \circ \varphi \simeq \psi \quad \text{and} \quad \mu \circ \psi \simeq \varphi \quad \text{and} \quad \mu \circ \beta \simeq \alpha
\]
(and in fact two of these homotopies are equalities).

Corollary 8.15. Let the following diagram
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{i} & & \downarrow{j} \\
X & \xrightarrow{g} & Y
\end{array}
\]
be a pushout in \( \text{Top} \) (or \( \text{Top}_* \)) in which \( i \) (and hence \( j \)) is a cofibration. If \( f \) is a homotopy equivalence, then so is \( g \).

Proof. Let \( e : B \rightarrow A \) be a homotopy inverse for \( f \), and form consecutive pushouts as in the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{i} & & \downarrow{j} \\
X & \xrightarrow{h} & Y & \xrightarrow{m} & Z & \xrightarrow{n} & W
\end{array}
\]
Then all the vertical maps are cofibrations. Now \( e \circ f \) and \( f \circ e \) are homotopic to the identities, so by Proposition 8.12 and Remark 8.14, there are homotopy equivalences \( \lambda : Z \xrightarrow{\simeq} X \) (and \( \lambda \circ k \simeq i \)), and \( \lambda' : W \xrightarrow{\simeq} Y \) with \( \lambda' \circ m \circ h \simeq \text{id}_X \) (and \( \lambda' \circ l \simeq j \)). Since \( \lambda \) and \( \lambda' \) are homotopy equivalences, so are \( m \circ h \) and \( n \circ m \). But then \( m \) itself must be a homotopy equivalence, by evident properties of isomorphisms in the homotopy category. Since \( \lambda \circ m \circ h \simeq \text{id}_X \) it then follows that \( h \) is a homotopy equivalence as well. \( \square \)
LECTURE 9: CELLULAR APPROXIMATION

The aim of this section is to give a proof of the cellular approximation theorem. One formulation of this theorem is that every map between CW complexes is homotopic to a cellular map (but we will also see a version for CW pairs). Recall that a map of CW complexes is cellular if it restricts to a map of n-skeleta for all n. Thus, such a cellular map does not increase the dimension of cells. The proof of this theorem is rather involved and will occupy the bulk of this section. The main work will be to establish Lemma 9.4 which covers a particular case. In the next section we will use the cellular approximation theorem in order to deduce the famous ‘Whitehead’s theorem’.

Before we begin with the cellular approximation theorem, let us recall a fact from point-set topology. In general, it is not true that the formation of quotient spaces and products would be compatible. More precisely, let \( q : X \to Y \) be a map exhibiting \( Y \) as a quotient of \( X \). In particular, a subset of \( Y \) is open if and only if the preimage \( q^{-1}(U) \) is open in \( X \). If \( Z \) is an arbitrary space, then, in general, we can not conclude that the map \( q \times \text{id}_Z : X \times Z \to Y \times Z \) is a quotient map. However, there is the following fact.

**Lemma 9.1.** Let \( q : X \to Y \) be a quotient map and let \( K \) be a compact space. Then also the map \( q \times \text{id}_K : X \times K \to Y \times K \) is a quotient map.

We will be particularly interested in the following situation. Let \( X \) be a CW complex. The \( n \)-skeleton \( X^{(n)} \) of \( X \) is obtained from the \((n-1)\)-skeleton by attaching a set of \( n \)-cells. Thus, we have a quotient map

\[
q_n : X^{(n-1)} \sqcup J_n \times e^n \longrightarrow X^{(n)}.
\]

If we form the product of this map with the identity of \( I = [0,1] \) then the previous lemma implies the following.

**Corollary 9.2.** Let \( X \) be a CW complex. Then there is a quotient map

\[
q_n \times \text{id}_I : X^{(n-1)} \times I \sqcup J_n \times e^n \times I \longrightarrow X^{(n)} \times I.
\]

Let us immediately give the statement of the cellular approximation theorem.

**Theorem 9.3** (Cellular approximation theorem). Let \((X,A)\) be a CW pair, let \( Y \) be a CW complexes, and let \( f : X \to Y \) be a map of spaces. If \( f|_A : A \to Y \) is cellular, then \( f \) is homotopic to a cellular map \( g : X \to Y \) relative to \( A \). In particular, any map of CW complexes is homotopic to a cellular one.

Since CW complexes are built inductively, the following strategy will not come as a surprise. Given a map \( f : X \to Y \) of CW complexes, we will try to deform \( f \) cell by cell into a cellular map. As an important building block for the proof of the theorem, there is the following case of a single cell.
Lemma 9.4. Let $Y$ be obtained from $B$ by attaching an $n$-cell, that is, assume that we have a pushout diagram of the form

$$
\begin{array}{c}
\partial D^n \\
\downarrow \\
D^n \\
\downarrow
\end{array}
\begin{array}{c}
\rightarrow B \\
\leftarrow \\
\rightarrow Y
\end{array}
$$

Then any map $f : (D^m, \partial D^m) \to (Y, B)$ with $m < n$ is homotopic relative to $\partial D^m$ to a map $g$ satisfying $g(D^m) \subseteq B$.

Proof. The proof of this lemma will be given at the end of this section.

Let us now use this lemma to give a proof of the cellular approximation theorem. As usual, given a CW complex $Y$, the skeleton filtration will be denoted by $Y^{(0)} \subseteq Y^{(1)} \subseteq Y^{(2)} \subseteq \ldots$, $n \in \mathbb{N}$.

Proof of Theorem 9.3 (using Lemma 9.4). Thus, we are given a map $g_{-1} = f : X \to Y$ which restricts to a cellular map on $A = X^{(-1)}$. We will inductively construct maps $g_n : X \to Y$ and homotopies $H_n : g_{n-1} \simeq g_n$ for $n \geq 0$ such that

(i) The map $g_n$ sends the relative $n$-cells, i.e., the ones given by the index set $J_n(X) - J_n(A)$, to $Y^{(n)}$.

(ii) The homotopy $H_n : g_{n-1} \simeq g_n$ is relative to $X^{(n-1)}$.

Recall that a CW pair is, in particular, a relative CW complex so that we have a filtration

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,$$

which has the following two properties:

(i) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching $n$-cells for $n \geq 0$.

(ii) The space $X$ is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology.

Let us assume inductively that we are already given the map $g_{n-1}$, and let us construct $g_n$ and $H_n$.

Now, denoting the set of relative $n$-cells by $J_n$, there is a pushout diagram

$$
\begin{array}{c}
J_n \times \partial e^n = \bigsqcup_{\sigma \in J_n} \partial e^n_\sigma \\
\downarrow \\
J_n \times e^n = \bigsqcup_{\sigma \in J_n} e^n_\sigma (\chi_*) \rightarrow X^{(n)}
\end{array}
$$

Let us assume that there are cells $\sigma \in J_n$ such that $g_{n-1}(e^n_\sigma)$ is not contained in $Y^{(n)}$ (otherwise we set $g_n = g_{n-1}$ and take $H_n$ to be the constant homotopy). For each such cell $e^n_\sigma$, there is a finite relative subcomplex $Y'$ with $Y^{(n)} \subseteq Y' \subseteq Y$ such that $g_{n-1}(e^n_\sigma) \subseteq Y'$. Take a cell of maximal dimension in $Y'$ which has a nontrivial intersection with $g_{n-1}(e^n_\sigma)$. Then Lemma 9.4 tells us that this cell can be avoided up to relative homotopy. Repeating this finitely many times and gluing the relative homotopies together, we obtain a homotopy $H_{n, \sigma} : g_{n-1} \simeq g_{n, \sigma} : e^n_\sigma \to Y$ relative to $\partial e^n_\sigma$ such that $g_{n, \sigma}(e^n_\sigma) \subseteq Y^{(n)}$. Recall from Corollary 9.2 that $X^{(n)} \times [0, 1]$ carries the quotient topology with respect to the map

$$X^{(n-1)} \times [0, 1] \sqcup J_n \times e^n \times [0, 1] \rightarrow X^{(n)} \times [0, 1].$$
Thus, we can glue the homotopies $H_n, \sigma$, the constant homotopies on $g_{n-1}: e^n_\sigma \to Y$ for all $n$-cells with $g_{n-1}(e^n_\sigma) \subseteq Y(n)$, and the constant homotopy on $g_{n-1}|_{X(n-1)}$ together in order to obtain a homotopy
\[ \tilde{H}_n: g_{n-1}|_{X(n)} \simeq \tilde{g}_n: X(n) \times [0, 1] \to Y. \]

From these data we can form the following extension problem
\[ X \times \{0\} \cup X(n) \times [0, 1] \xrightarrow{(g_{n-1}, \tilde{H}_n)} Y \]
\[ X \times [0, 1] \xrightarrow{\exists H_n} \]
which admits a solution since the inclusion $X(n) \to X$ is a cofibration. It only remains to set $g_n = H_n(-, 1)$ in order to conclude the inductive step.

If the CW complex $X$ is finite-dimensional, i.e., if $X(n) = X$ for some $n \geq 0$, then we are done since it suffices to compose the finitely many homotopies $H_k$, $0 \leq k \leq n$, to obtain a homotopy $H: f \simeq g = g_n$ relative to $A$ such that $g: X \to Y$ is a cellular map.

For an infinite-dimensional CW complex we can conclude by the following argument. In that case we have to check that these infinitely many homotopies can be assembled into a single homotopy $H: X \times I \to Y$. In fact, as the homotopies $H_n$ are relative to $X(n-1)$, it follows that $H_k$ is stationary on $X(n-1)$ for $k \geq n$. Thus, we define $H$ on $X(n-1)$ by first running through $H_0$ at a double speed, then through $H_1$ at a fourfold speed, through $H_2$ at an 8-fold speed, and so on. After having run through $H_{n-1}$, the map $H|_{X(n-1)}$ is defined to be stationary. We leave it to the reader to check that this way we obtain a continuous map $H: X \times I \to Y$. From the definition it is immediate that $H$ is a homotopy relative to $A$ such that $g = H(-, 1): X \to Y$ is a cellular map as intended. \hfill \Box

**Corollary 9.5.** The homotopy groups $\pi_k(S^n, *)$ are trivial for all $1 \leq k < n$.  

**Proof.** Let us endow $S^n$ with the CW structure consisting of a unique 0-cell and a unique $n$-cell. By Theorem 9.3, any map $f: S^k \to S^n$ is homotopic to a cellular map $g: S^k \to S^n$. But, for $k < n$ we have that the $k$-skeleton of $S^k$ is the entire $k$-sphere, while the $k$-skeleton of $S^n$ consists of a point only. Thus, $g$ is a constant map and we are done. \hfill \Box

We will see more applications of the cellular approximation theorem as the course goes on. In the remainder of this section we give a proof of Lemma 9.4.

**Proof of Lemma 9.4 (not using Theorem 9.3).** The proof will be given by induction over $n$, the dimension of the cell attached to $B$. Let us first establish the case of $n = 1$. Thus, $m = 0$ and hence $\partial D^m = \emptyset$ and $D^m = *$. A map $f = \kappa_y: (*, 0) \to (Y, B)$ is essentially the same as a point $y$ in $Y$. There is a path $\omega: I \to Y$ with $\omega(0) = y$ and $\omega(1) = b \in B$. This path defines the desired homotopy $f = \kappa_y \simeq \kappa_b = g$.

Before performing the induction step, let us describe the strategy of the proof. The attaching map $\chi: D^m \to Y$ restricts to a homeomorphism from the interior of the disc onto its image in $Y$. The main work consists in showing that we can construct a homotopy $f \simeq h$ relative to the boundary $\partial D^m$ such that $h$ omits the origin of the interior of the attached disc. To see that this is enough, let us denote by $Y - \{o\}$ the space which we obtain from $Y$ by removing that origin. It is easy to see that $i: B \to Y - \{o\}$ is the inclusion of a strong deformation retraction (induced by
collapsing the punctured $n$-disc $D^n - \{o\}$ onto $S^{n-1}$). Part of this strong deformation retraction is a homotopy

$$\text{id}_{Y - \{o\}} \simeq i \circ r \quad \text{relative to } B$$

which induces the desired relative homotopy $h = \text{id} \circ h \simeq i \circ r \circ h = g$ relative to $\partial D^m$. Putting these two homotopies together we conclude that $f \simeq g$ relative to $\partial D^m$ as intended.

Let us now assume inductively that we already established the lemma for $n - 1$. Before attacking the induction step, we list the following consequences of our inductive assumption:

(i) Any map $S^k \to S^{n-1}$ as well as any map $S^k \to S^{n-1} \times (a, b)$ for $k < n - 1$ is homotopic to a constant map.

(ii) Any map $S^k \to S^{n-1} \times (a, b)$, $k < n - 1$, can be extended to a map on $D^{k+1}$.

In fact, the first case of the first point is established by an application of the lemma for $S^k \to S^{n-1} \times (a, b)$ which corresponds to our given map $S^k \to S^{n-1} \times (a, b)$ for $k < n - 1$ is homotopic to a constant map and only if it can be extended over the cone of its domain. Thus, it remains to construct a homotopy $f \simeq h: D^m \to Y$ relative to $\partial D^m$ such that $h$ does not hit the origin. This will be done by a rather elaborate application of the lemma of Lebesgue to an adapted open cover of $D^m$. We begin by constructing an open cover of $Y$. For this purpose, let us introduce notations for the subsets

$$U' = \{ x \in D^m \mid 0 \leq \|x\| < 2/3 \} \quad \text{and} \quad V' = \{ x \in D^m \mid \|x\| > 1/3 \}$$

and let us define two subsets of $Y$ by setting

$$U = \chi(U') \quad \text{and} \quad V = B \cup_{\partial D^m} \chi(V')$$

where $\chi: D^m \to Y$ is the characteristic map of the $n$-cell. The aim is to construct a relative homotopy $f \simeq h$ such that the image of $h$ entirely lies in $V$, and hence, in particular, avoids the point $o \in Y$.

By construction, $U$ and $V$ define an open cover of $Y$. Since $\chi$ induces a homeomorphism when restricted to the interior of the $n$-disc, we obtain a homeomorphism

$$U \cap V \cong S^{n-1} \times (1/3, 2/3)$$

so that we can later apply the second consequence above to $U \cap V$.

Using our favorite homeomorphism of pairs $(I^m, \partial I^m) \cong (D^m, \partial D^m)$, we are thus in the following situation:

$$
\begin{array}{ccc}
I^m & \simeq \rightarrow & D^m \\
\downarrow & & \downarrow f \\
\partial I^m & \cong \rightarrow & S^{m-1} \\
\downarrow & & \downarrow f \\
B
\end{array}
$$

Pulling back the open cover of $Y$ along $f$ induces an open cover $f^{-1}(U)$, $f^{-1}(V)$ of the compact metric space $[0,1]^m$. The lemma of Lebesgue guarantees the existence of a natural number $N > 0$ such that the image of each $m$-cube

$$I^m_{k_1, \ldots, k_m} = [k_1/N, (k_1 + 1)/N] \times \cdots \times [k_m/N, (k_m + 1)/N], \quad 0 \leq k_i < N,$$
under \( f \) lies in \( U \) or in \( V \). We now want to construct relative homotopies to modify \( f \) on those sub-cubes of the \( I_{k_1,...,k_m}^m \) which are not entirely mapped to \( V \) while we want to keep it unchanged on the remaining sub-cubes.

For this purpose, let us define a filtration on \( X = I^m \),

\[
\partial I^m \subseteq X^{(-1)} \subseteq X^{(0)} \subseteq \ldots \subseteq X^{(m)} = X = I^m,
\]
as follows. Let \( J_{-1} \) be an index set for all \( l \)-sub-cubes, \( 0 \leq l \leq m \), of \( I_{k_1,...,k_m}^m \), \( 0 \leq k_i < N \), which are already completely mapped to \( V \) by \( f \). Let us denote the \( l \)-sub-cube corresponding to such an index \( \phi \in J_{-1} \) by \( I^l_\phi \). We then set

\[
X^{(-1)} = \bigcup_{\phi \in J_{-1}} I^l_\phi,
\]
and it follows from our assumption on \( f \) that \( \partial I^m \subseteq X^{(-1)} \). We now have to take care of the remaining sub-cubes and this will be done by induction over the dimension of these sub-cubes. Thus, for each \( 0 \leq k \leq m \), let \( J_k = \{ \phi \} \) be an index set for all \( k \)-dimensional sub-cubes \( I^k_\phi \) of the cubes \( I_{k_1,...,k_m}^m \) which satisfy \( f(I^k_\phi) \not\subseteq V \). We then inductively set

\[
X^{(k)} = X^{(k-1)} \cup \bigcup_{\phi \in J_k} I^k_\phi.
\]

By definition, this gives us an exhaustive filtration of \( X = I^m \) (which, in fact, defines a relative CW complex \((X, X^{(-1)})\)).

We now want to inductively construct maps \( h_k : X^{(k)} \to Y, \ k \geq -1 \), such that:

(i) The map \( h_{-1} \) is obtained from \( f \) by restriction.

(ii) The map \( h_k \) sends the cubes \( I^k_\phi \) to \( U \cap V \) for all \( \phi \in J_k \) and \( k \geq 0 \).

(iii) The map \( h_k \) extends \( h_{k-1} \), i.e., we have \( h_k |_{X^{(k-1)}} = h_{k-1} \) for all \( k \geq 0 \).

For \( h_0 \), note that \( X^{(0)} \) is obtained from \( X^{(-1)} \) by possibly adding some vertices which are mapped to \( U \). For each such vertex, choose a path to a point in \( U \cap V \). These target points together with \( h_{-1} \) then define the map \( h_0 \). For the inductive step, let us assume that \( h_{k-1} \) has already been constructed. For each \( \phi \in J_k \), we have that \( h_{k-1}(\partial I^k_\phi) \subseteq U \cap V \) (there are two different arguments for the two types of faces, one of them using that \( N \in \mathbb{N} \) was chosen to be adapted to \( f^{-1}(U) \) and \( f^{-1}(V) \)). Recall that by construction we have a homeomorphism \( U \cap V \cong S^{n-1} \times (1/3, 2/3) \) so that we can our inductive assumption to find extensions as indicated in the following diagram:

\[
\begin{array}{c}
\partial I^k_\phi \xrightarrow{h_{k-1}} U \cap V \\
I^k_\phi \xrightarrow{h_{k,\phi}} \end{array}
\]

It is easy to see that these maps \( h_{k,\phi} \) and \( h_{k-1} \) can be assembled together in order to define a map \( h_k : X^{(k)} \to Y \) with the desired properties. If we set \( h = h_m : I^m = X^{(m)} \to Y \) then we have \( h(I^m) \subseteq V \). Hence it suffices to show that \( f \simeq h \) relative to \( \partial I^m \).

We will in fact show that we can construct such a homotopy relative to \( X^{(-1)} \). By construction, both maps \( f \) and \( h \) coincide on \( X^{(-1)} \). Moreover, the restrictions of both maps to \( X \setminus X^{(-1)} \) can be considered as maps taking values in \( U \). But, \( U \) is homeomorphic to an open \( n \)-disc, hence convex, so that the two restrictions are homotopic via linear homotopies. This homotopy together with
the constant homotopy on $X^{(-1)}$ can be assembled together to give the desired homotopy $f \simeq h$ relative to $X^{(-1)}$ concluding the proof. \qed
LECTURE 10: CW APPROXIMATION AND WHITEHEAD’S THEOREM

In this section we will establish the two important theorems showing up in the title. The first of them, the theorem on the existence of CW approximations (Theorem 10.7), emphasizes the importance of CW complexes: up to weak equivalence any space can be replaced by a CW complex. Thus, if one is only interested in spaces up to this notion of equivalence, then it is enough to deal with CW complexes. The second theorem, the celebrated Whitehead theorem (Theorem 10.17), tells us that CW complexes are better behaved than arbitrary spaces in the following sense. The notions of weak homotopy equivalence and (actual) homotopy equivalence coincide if we only consider maps between CW complexes.

In both theorems the notion of a weak homotopy equivalence plays a key role so let us begin by introducing that concept.

**Definition 10.1.** A map of spaces \( f : X \rightarrow Y \) is a weak homotopy equivalence if the induced maps
\[
f_* : \pi_k(X, x_0) \rightarrow \pi_k(Y, f(x_0))
\]
are bijections for all dimensions \( k \) and all base points \( x_0 \in X \).

Note that we insist that we have isomorphisms of homotopy groups for all points \( x_0 \in X \). If one weakens this condition by considering a single base point only, then one obtains a different notion which we do not want to axiomatize here. The good notion is the one given above. Of course, the motivation for the terminology stems from the first point in the following exercise.

**Exercise 10.2.**

(i) Let \( f : X \rightarrow Y \) be a homotopy equivalence. Then \( f \) is a weak equivalence. (Note that the functoriality of the homotopy groups does not suffice to solve this part!)

(ii) Let \( f : X \rightarrow Y, g : Y \rightarrow Z \) be maps of spaces, and let \( h = gf : X \rightarrow Z \) be their composition. Show that if two of the maps \( f, g, \) and \( h \) are weak equivalences then so is the third one.

(iii) Two spaces \( X \) and \( Y \) are called weakly equivalent if there are finitely many weak equivalences
\[
X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \ldots \longleftarrow X_{n-1} \longrightarrow X_n = Y
\]
pointing possibly in different directions which ‘connect’ \( X \) and \( Y \). Check that this is an equivalence relation. The equivalence classes with respect to this equivalence relation are called weak homotopy types.

(iv) More generally, consider a relation \( R \subseteq S \times S \). Define explicitly the equivalence relation \( \sim_R \) on \( S \) generated by \( R \), that is, the smallest equivalence relation which contains \( R \). Relate this to the previous part of the exercise (ignore set-theoretical issues for this comparison!).

There are the following classes of maps which allow us to measure how far a map is from being a weak equivalence.

**Definition 10.3.** Let \( f : X \rightarrow Y \) be a map of spaces and let \( n \geq 0 \). Then \( f \) is an \( n \)-equivalence if for all \( x_0 \in X \) the induced map
\[
f_* : \pi_k(X, x_0) \rightarrow \pi_k(Y, f(x_0))
\]
is bijective for \( k \leq n - 1 \) and surjective for \( k = n \).
Thus, a map of spaces is a weak equivalence if and only if it is an \( n \)-equivalence for all \( n \geq 0 \). An interesting class of examples for this notion is given by the inclusions which are part of the skeleton filtration of a CW complex.

**Lemma 10.4.** Let \( X \) be a CW complex and let \( i_n : X^{(n)} \to X \) be the inclusion of the \( n \)-skeleton. Then \( i_n \) is an \( n \)-equivalence.

**Proof.** This follows from a repeated application of the cellular approximation theorem. In order to obtain the surjectivity, consider a class \( \alpha \in \pi_k(X,*) \) which can be represented by a cellular map \( S^k \to X \). Thus, for all \( k \leq n \), we can find a representative which factors over \( i_n : X^{(n)} \to X \) showing that \( \alpha \) lies in the image of \( i_n : \pi_k(X^{(n)},*) \to \pi_k(X,*) \).

For the injectivity, consider two classes \( \alpha, \beta \in \pi_k(X^{(n)},*) \) for \( k < n \), and represent them by cellular maps \( f : S^k \to X^{(n)} \) and \( g : S^k \to X^{(n)} \) respectively. By assumption, we can find a homotopy
\[
H : S^k \times I \to X,
\]
\[
H : f \simeq g.
\]
Since \( I \) is compact, the space \( S^k \times I \) is a CW complex, and, by the explicit description of the CW structure, the subspace \( S^k \times \partial I \) is a subcomplex. Now, the homotopy is a map which is already cellular on this subcomplex. Thus, an application of the cellular approximation theorem implies that we can find a cellular map \( H' : S^k \times I \to X \) which restricts to \( f \) and \( g \) on the boundary components. Thus, this map factors over the inclusion \( i_n \) showing that \( \alpha = \beta \) as intended. \( \square \)

**Exercise 10.5.** Let \( (X,x_0) \) be a pointed, connected space, \( Y \) an arbitrary space, and \( n \geq 0 \). Then a map \( f : X \to Y \) is an \( n \)-equivalence if and only if the induced map
\[
f_* : \pi_k(X,x_0) \to \pi_k(Y,f(x_0))
\]
is bijective for \( k \leq n-1 \) and surjective for \( k = n \).

1. **CW approximation**

Let us now show that up to weak equivalence every topological space is a CW complex.

**Definition 10.6.** A **CW approximation** of a topological space \( X \) is a CW complex \( K \) together with a weak equivalence \( f : K \to X \).

**Theorem 10.7** (Existence of CW approximations). *Every space has a CW approximation.*

**Proof.** Let \( X \) be an arbitrary space. We can assume that the space \( X \) is path-connected by constructing a CW approximation for each path-component separately. It is easy to see that these CW approximations then assemble to one for the entire space.

We will now give an inductive construction of a CW approximation of \( X \). More precisely, we will first construct \( n \)-equivalences
\[
f_n : K_n \to X,
\]
for certain \( n \)-dimensional CW complexes \( K_n \) and then show that these maps can be assembled to a CW approximation \( f : K \to X \).

In dimension \( n = 0 \) we let \( K_0 = * \) be a single point and let \( f_0 : K_0 \to X \) be the inclusion of an arbitrary point of \( X \) which obviously is a 0-equivalence. Let us assume inductively that we have already constructed an \( n \)-equivalence \( f_n : K_n \to X \) with \( K_n \) an \( n \)-dimensional CW complex. We will construct the map \( f_{n+1} \) in two steps. First let us take care of the possibly non-trivial kernel
\[
A_n = \ker (f_n : \pi_n(K_n,*) \to \pi_n(X,*)).
\]
Choose an arbitrary set of generators \((a_\sigma)_{\sigma \in J'_n}\) for the group \(A_n\). Each generator can be represented by a map \(\chi_\sigma : \partial e^{n+1} \to K_n\), and by definition of \(A_n\) we can choose homotopies \(H_{n,\sigma}\) from \(f_n \circ \chi_\sigma\) to a constant map. Now, construct the intermediate space \(K'_{n+1}\) by attaching \((n+1)\)-cells to \(K_n\) as follows:

\[
\begin{array}{ccc}
J''_{n+1} \times \partial e^{n+1} & \xrightarrow{\chi_\sigma} & K_n \\
\downarrow & & \downarrow \\
J''_{n+1} \times e^{n+1} & \xrightarrow{\psi_n} & K'_{n+1}
\end{array}
\]

This way we obtain an \((n+1)\)-dimensional CW complex \(K'_{n+1}\) such that \(i'_n : K_n \to K'_{n+1}\) is the inclusion of the \(n\)-skeleton. Since \(K'_{n+1}\) is endowed with the quotient topology, it is easy to see that the homotopies \(H_{n,\sigma}\) and the map \(f_n\) together induce a map \(f'_{n+1} : K'_{n+1} \to X\) such that \(f'_{n+1} \circ i'_n = f_n\). By Lemma 10.4 we know that \(i'_n\) is an \(n\)-equivalence, as is \(f_n\) by inductive assumption so that the same is also true for \(f'_{n+1}\). Moreover, we can use the cellular approximation theorem to conclude that the induced map

\[f'_{n+1} : \pi_n(K'_{n+1}, \ast) \to \pi_n(X, \ast)\]

is also injective. In fact, given an element \(\alpha'\) in the kernel of that map, then there is a cellular map \(S^n \to K'_{n+1}\) representing that class which hence factors as \(S^n \to K_n \to K'_{n+1}\). We leave it to the reader to conclude from here that \(\alpha'\) is trivial.

We next address the problem that the induced map might not be surjective in dimension \(n + 1\). Thus, let us consider the possibly non-trivial cokernel

\[B_{n+1} = \text{coker}(f'_{n+1} : \pi_n(K'_{n+1}, \ast) \to \pi_{n+1}(X, \ast))\]

and let \((b_\sigma)_{\sigma \in J''_{n+1}}\) be a set of generators of \(B_{n+1}\). Define \(K_{n+1}\) to be the wedge

\[K_{n+1} = K'_{n+1} \vee \bigvee_{\sigma \in J''_{n+1}} S^{n+1}.
\]

Alternatively, this can also be described as an attachment of \((n+1)\)-cells using constant attaching maps, that is, we have a pushout diagram

\[
\begin{array}{ccc}
J''_{n+1} \times \partial e^{n+1} & \xrightarrow{(\kappa_\sigma)} & K'_{n+1} \\
\downarrow & & \downarrow \\
J''_{n+1} \times e^{n+1} & \xrightarrow{\xi_n} & K_{n+1}
\end{array}
\]

In both descriptions (using the usual homeomorphism \(e^{n+1}/\partial e^{n+1} \cong \partial e^{n+2}\) in the second one), the generators \(b_\sigma\) together with the map \(f'_{n+1}\) can be assembled to define a map \(f_{n+1} : K_{n+1} \to X\) which satisfies \(f_{n+1} \circ i'_n = f'_{n+1} : K'_{n+1} \to X\) and hence

\[f_{n+1} \circ i_n = f_n : K_n \to X,
\]

where \(i_n = i''_n \circ i'_n : K_n \to K'_{n+1} \to K_{n+1}\). We leave it to the reader to check that \(f_{n+1}\) is an \((n+1)\)-equivalence.

Thus, we have constructed \(n\)-dimensional CW complexes \(K_n\) together with \(n\)-equivalences \(f_n\) and inclusions \(i_n : K_n \to K_{n+1}\) which are compatible with the \(n\)-equivalences. Let us denote by \(K\) the union \(\bigcup_n K_n\) endowed with the weak topology. Then it is easy to see that \(K\) is a CW complex.
(with a single 0-cell and the set of \(n\)-cells given by \(J_n = J'_n \cup J''_n\) for \(n \geq 1\)) such that its \(n\)-skeleton is given by \(K^{(n)} = K_n\). The maps \(f_n\) induce a unique map \(f: K \to X\) such that \(f|_{K_n} = f_n\). The final claim is that \(f\) is a weak equivalence which we also leave to the reader (a further application of the cellular approximation theorem!).

\[\Box\]

**Exercise 10.8.** Conclude the proof of Theorem 10.7 by establishing the following three steps (we use the notation of the proof):

(i) The map \(f'_{n+1}: \pi_n(K'_{n+1}, *) \to \pi_n(X, *)\) is injective (and hence a bijection).

(ii) The map \(f_{n+1}: K_{n+1} \to X\) constructed in the induction step is an \((n+1)\)-equivalence.

(iii) The map \(f: K \to X\) is a weak equivalence.

**Remark 10.9.** We thus showed that every space is up to weak homotopy equivalence a CW complex. One might wonder if there is a functorial way of doing this. The first step would consist of the following problem. Let \(X \to Y\) be a map of spaces and let \(K \to X\) and \(L \to Y\) be CW approximations of \(X\) and \(Y\) respectively. Can we then find a map \(K \to L\) such that the following diagram commutes:

\[
\begin{array}{ccc}
K & \longrightarrow & X \\
\downarrow & \searrow & \downarrow \\
L & \longrightarrow & Y \\
\end{array}
\]

The first partially affirmative answer to this question (which lies only slightly beyond the scope of this course) is the following: we can always achieve this if we only insist that the square commutes up to homotopy; that is, if we are asking for the existence of such a map such that both compositions are homotopic.

The second affirmative answer is even more positive. A construction of such a functorial CW approximation can be given using ‘simplicial methods’. Given a space \(X\) one would consider all maps \(\Delta^n \to X\) for the various \(n \geq 0\) where \(\Delta^n\) is the geometric \(n\)-simplex, i.e., the convex hull of the \(n+1\) standard basis vectors of \(\mathbb{R}^{n+1}\). For each \(n\), one can single out a suitable subset \(J_n \subseteq \text{hom}_{\text{top}}(\Delta^n, X)\) such that these sets serve as index sets for a suitable CW complex. It can then be shown that these CW complexes are part of functorial CW approximation.

The ‘simplicial methods’ alluded to in the second affirmative answer are very powerful and show up in many areas of mathematics. In particular, the so-called simplicial sets –introduced in the 1950’s– provide an interesting, purely combinatorial approach to homotopy theory whose importance in modern homotopy theory (and in other areas of mathematics) can hardly be overestimated.

The mapping cylinder construction allowed us in a previous lecture to show that every map can be factored into a cofibration followed by a strong deformation retraction. We would like to have a refinement of this result for the case of a cellular map between CW complexes. Recall that the mapping cylinder \(M_f\) of a map \(f\) is given by the following pushout construction:

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & \searrow & \downarrow \\
X \times I & \longrightarrow & M_f \\
\end{array}
\]

**Proposition 10.10.** The mapping cylinder \(M_f\) of a cellular map \(f: X \to Y\) is again a CW complex which contains both \(X\) and \(Y\) as subcomplexes.
Proof. We will not give a proof of this result but instead refer the reader to the book ‘Cellular structures in topology’ by Rudolf Fritsch and Renzo Piccinini. □

With this preparation we now obtain the following refinement of the factorization result.

Corollary 10.11. Any cellular map can be factored as the inclusion of a CW subcomplex followed by a strong deformation retraction.

2. Whitehead’s theorem

A space is path-connected if any two points can be connected by a path, that is, if \( \pi_0 \) applied to the space gives us a one-point set. Similarly, a space is called simply connected, if it is path-connected and has a trivial fundamental group (by the action of the fundamental groupoid it is not important which base point we consider). Let us generalize these definitions to higher dimensions.

Definition 10.12. A space \( X \) is \( n \)-connected if \( \pi_k(X,x_0) \cong * \) for all \( k \leq n \) and all \( x_0 \in X \).

There is also a variant for pairs of spaces \((X,A)\). Given an arbitrary point \( a_0 \in A \) we gave a definition of \( \pi_n(X,A,a_0) = \pi_n(X,A) \) in the case that \( n \geq 1 \). For \( n \geq 2 \) these are naturally groups which are abelian if \( n \geq 3 \). In fact, the definition of the underlying pointed set of \( \pi_n(X,A) \) was as the set of homotopy classes of maps of triples
\[
\pi_n(X,A) = \left[ (I^n, \partial I^n, J^{n-1}), (X,A,a_0) \right]
\]
where \( J^{n-1} = I^{n-1} \times \{0\} \cup \partial I^{n-1} \times I \subseteq \partial I^n \subseteq I^n \). There are homeomorphisms \( I^n/J^{n-1} \cong D^n \) and \( \partial I^n/J^{n-1} \cong S^{n-1} \), and using these it is easy to show that we have natural bijections
\[
\pi_n(X,A) \cong \left[ (D^n, S^{n-1}, *), (X,A,a_0) \right].
\]
Motivated by the long exact homotopy sequence of a pointed pair, let us say that \( \pi_0(X,A) \cong * \) if the map \( \pi_0(A,a_0) \to \pi_0(X,a_0) \) is surjective, i.e., if each path-component of \( X \) has a non-trivial intersection with \( A \).

Definition 10.13. A pair of spaces \((X,A)\) is \( n \)-connected if \( \pi_k(X,A,a_0) \cong * \) for all \( k \leq n \) and for all \( a_0 \in A \).

We leave it as an exercise to establish the equivalence of the following statements.

Exercise 10.14. Let \((X,A)\) be a pair of spaces and let \( n \geq 0 \). Then the following are equivalent:

(i) Every map \( (D^n, S^{n-1}) \to (X,A) \) is homotopic to \( S^{n-1} \) to a map \( D^n \to A \).

(ii) Every map \( (D^n, S^{n-1}) \to (X,A) \) is homotopic through such maps to a map \( D^n \to A \).

(iii) Every map \( (D^n, S^{n-1}) \to (X,A) \) is homotopic through such maps to a constant map.

(iv) We have \( \pi_n(X,A,a_0) = \pi_n(X,A) \cong 0 \) for all \( a_0 \in A \).

This exercise is the basic building block for the following lemma which in turn is the key step towards Whitehead’s theorem.

Lemma 10.15. Let \((X,A)\) be a relative CW complex and let \((Y,B)\) be a pair of spaces with \( B \neq \emptyset \) and such that \( \pi_n(Y,B) = 0 \) for all dimensions such that \( X - A \) has \( n \)-cells. Then any map \( f: (X,A) \to (Y,B) \) is homotopic relative \( A \) to a map with image in \( B \).

Proof. By assumption we have a filtration of \( X \),
\[
A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,
\]
such that the following two properties are satisfied:
(i) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching $n$-cells for $n \geq 0$.
(ii) The space $X$ is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology and hence comes, in particular, with continuous inclusions $i_n: X^{(n)} \to X$.

The plan is to inductively construct intermediate maps $g_n: X \to Y$ such that $g_n(X^{(n)}) \subseteq B$ and homotopies $H_n: g_n \simeq g_{n-1}$ relative to $X^{(n-1)}$. We will then show how to conclude from here. By assumption the restriction $f_{-1}: A = X^{(-1)} \to Y$ already satisfies $f_{-1}(A) \subseteq B$ so that we set $g_{-1} = f_{-1}$.

Now let us assume inductively that maps $g_k$ and homotopies $H_k$ have already been constructed for $k < n$ and let $X - A$ have $n$-cells (otherwise the induction step is trivial) which we then index by a set $J_n$. For each $\sigma \in J_n$, let $\chi_\sigma: e^n \to X$ be an attaching map of the cell so that the square on the left is a pushout diagram:

$$
\begin{array}{c}
J_n \times \partial e^n \xrightarrow{(\chi_\sigma)} X^{(n-1)} \\
J_n \times e^n \xrightarrow{(\chi_\sigma)} X^{(n)}
\end{array}
\xrightarrow{g_{n-1}} Y
$$

Now, each characteristic map induces an element $g_{n-1} \circ \chi_\sigma: (e^n, \partial e^n) \to (Y, B)$. Since by assumption $\pi_n(Y, B) \cong 0$ we can use Exercise 10.14 to obtain a homotopy $\tilde{H}_{n,\sigma}: e^n \times I \to Y$ relative $\partial e^n$ from $g_{n-1} \circ \chi_\sigma: e^n \to Y$ to a map $g_{n,\sigma}$ which factors as $e^n \to B \to Y$. These homotopies together with the constant homotopy of $g_{n-1}$ can be assembled to define a homotopy

$$
\tilde{H}_n: X^{(n)} \times I \to Y: g_{n-1} \simeq g_n \text{ relative to } X^{(n-1)} \text{ with } g_n(X^{(n)}) \subseteq B.
$$

Since the inclusion $i: X^{(n)} \to X$ is a cofibration, we can find a lift in the following diagram

$$
\begin{array}{c}
(g_{n-1}, \tilde{H}_n): X \times \{0\} \cup X^{(n)} \times I \xrightarrow{\tilde{H}_n} Y \\
X \times I \xrightarrow{\tilde{H}_n}
\end{array}
$$

Setting $g_n = H_n(-, 1): X \to Y$ concludes the inductive step of the construction.

It remains to check that these infinitely many homotopies can be assembled into a single homotopy $H: X \times I \to Y$. In fact, as the homotopies $H_k$ are relative to $X^{(n-1)}$, it follows that $H_k$ is stationary on $X^{(n-1)}$ for $k \geq n$. Thus, we define $H$ on $X^{(n-1)}$ by first running through $H_0$ at a double speed, then through $H_1$ at a fourfold speed, etc. We leave it to the reader to check that this way we obtain a continuous map $H: X \times I \to Y$. From the definition it is immediate that $H$ is a homotopy relative to $A$ and such that $H(X, 1) \subseteq B$ as intended. $\square$

As an immediate consequence of this we collect the following convenient result.

**Corollary 10.16.** Let $(X, A)$ be a relative CW complex such that the inclusion $i: A \to X$ is a weak homotopy equivalence. Then $i$ is the inclusion of a strong deformation retract.

**Proof.** Apply the lemma to the identity morphism of $(X, A)$. $\square$

We can now use this lemma to establish the celebrated ‘Whitehead’s theorem’.
Theorem 10.17 (Whitehead’s theorem). Let \( f : X \to Y \) be a weak equivalence between CW complexes \( X \) and \( Y \). Then \( f \) is a homotopy equivalence. If \( f \) is the inclusion of a CW complex, then \( f \) is the inclusion of a strong deformation retract.

**Proof.** We leave it to the reader to reduce to the case of a path-connected CW complex. By the cellular approximation theorem we can assume that \( f \) is a cellular map. Moreover, the mapping cylinder construction allows us to assume that \( f = i : X \to Y \) is the inclusion of a subcomplex. The long exact sequence of homotopy groups of the pair \((Y, X)\) implies that all relative homotopy groups \( \pi_n(Y, X) \) vanish. Thus the previous lemma applied to the identity \( \text{id}: (Y, X) \to (Y, X) \) implies that \( \text{id} \simeq i \circ r \) relative to \( X \) for some map \( r : Y \to X \). Thus, this map \( r \) satisfies \( r|_X = i \), that is, \( r \circ i = \text{id}_X \). It follows that the map \( i : X \to Y \) is the inclusion of a strong deformation retract, hence, in particular, a homotopy equivalence. \( \square \)

Thus, from the knowledge about the behavior of a map at the level of homotopy groups we can actually *construct* a map in the converse direction. This indicates that the collection of invariants given by the homotopy groups at all points is very powerful.

**Remark 10.18.** Note however that Whitehead’s theorem does *not* imply that two CW complexes \( X \) and \( Y \) are homotopy equivalent as soon as the corresponding homotopy groups \( \pi_n(X) \) and \( \pi_n(Y) \) are isomorphic for all \( n \geq 0 \). To put it differently, it does not suffice to have abstract isomorphisms of these groups. Instead, it is essential that these isomorphisms are—at least in one direction—induced by an actual map of spaces.

A close inspection of the proof of Theorem 10.7 shows that we also have the following refined version.

**Corollary 10.19.** Let \( X \) be a \( n \)-connected space. Then there is a CW approximation \( K \to X \) such that \( K \) has a trivial \( n \)-skeleton, i.e., such that \( K^{(n)} = \ast \).

A combination of this corollary with Whitehead’s theorem gives the following nice fact.

**Corollary 10.20.** A \( n \)-connected CW complex is homotopy equivalent to a CW complex with trivial \( n \)-skeleton.

In the exercises, you will be asked to proof these two results. Using similar methods as above, one can also establish the following relative version of Whitehead’s theorem.

**Theorem 10.21** (relative version of Whitehead’s theorem). Let \( f : (X, A) \to (Y, B) \) be a weak equivalence of relative CW complexes such that \( f : A \to B \) is a homotopy equivalence. Then \( f : (X, A) \to (Y, B) \) is a homotopy equivalence of pairs.
LECTURE 11: POSTNIKOV AND WHITEHEAD TOWERS

In the previous section we used the technique of adjoining cells in order to construct CW approximations for arbitrary spaces. Here we will see that the same technique allows us to modify spaces by killing all homotopy groups above a certain dimension. This will be used to ‘approximate’ a connected space by a tower of spaces which have only non-trivial homotopy groups below or above a fixed dimension where they are isomorphic to the ones of the given space. The first case gives rise to the Postnikov tower and the second one to the Whitehead tower. Moreover, the homotopy groups of two subsequent levels in these towers only differ in one dimension. In fact, the maps belonging to the towers are fibrations and the fibers have precisely one non-trivial homotopy group.

1. The Postnikov tower

We know that if \( \alpha: \partial e^{n+1} \to X \) represents an element \([\alpha] \in \pi_n(X, x_0)\), then \([\alpha] = 0\) if and only if \(\alpha\) extends to a map \(e^{n+1} \to X\). Thus if we enlarge \(X\) to a space \(X' = X \cup_{\alpha} e^{n+1}\) by adjoining an \((n+1)\)-cell with \(\alpha\) as attaching map, then the inclusion \(i: X \to X'\) induces a map \(i_*: \pi_n(X, x_0) \to \pi_n(X', x_0)\) with \(i_*[\alpha] = 0\). We say that \([\alpha]\) ‘has been killed’. (Naively, we think of \(X'\) as a smallest extension of \(X\) that does that. Some justification for this thinking will be provided in the exercises.) The following lemma expresses what happens to the homotopy groups in lower dimensions. The proof is similar to the one that the inclusion of the \(n\)-skeleton of a CW complex is an \(n\)-equivalence and will hence not be given.

**Lemma 11.1.** Let \((X, x_0)\) be a pointed space, and let \(X' = X \cup_{\alpha} e^{n+1}\) be obtained from \(X\) by adjoining an \((n+1)\)-cell. Then the inclusion \(i: X \to X'\) induces a map \(\pi_k(X, x_0) \to \pi_k(X', x_0)\) which is an isomorphism for \(k < n\) and surjective for \(k = n\).

It is difficult to control what happens to the higher homotopy groups. For example, since \(\pi_3(S^2)\) is non-trivial, adding a 2-cell to an element in \(\pi_1\) may well add elements in \(\pi_3\). However, we can ‘kill’ all of \(\pi_n\) without changing \(\pi_k\) for \(k < n\), by iterating the procedure of Lemma 11.1.

**Lemma 11.2.** Let \((X, x_0)\) be a pointed space. Then there exists a relative CW complex \(i: X \to Y\), constructed by adjoining \((n+1)\)-cells only, such that \(i_*: \pi_k(X, x_0) \to \pi_k(Y, y_0)\) is bijective for \(k < n\) and such that \(\pi_n(Y, y_0) = 0\).

**Proof.** Let \(A\) be a set of representatives \(\alpha\) of generators \([\alpha]\) of the group \(\pi_n(X, x_0)\). Let \(Y\) be obtained from \(X\) by attaching an \((n+1)\)-cell \(e^{n+1}_\alpha\) along \(\alpha: \partial e^{n+1}_\alpha \to X\) for each \(\alpha \in A:\)

\[
\begin{array}{ccc}
A \times \partial e^{n+1} & \rightarrow & X \\
\downarrow & & \downarrow i \\
A \times e^{n+1} & \rightarrow & Y.
\end{array}
\]

Then by an iterated application of Lemma 11.1, the map \(i: X \to Y\) induces isomorphisms in \(\pi_k\) for \(k < n\), and induces the zero-map on \(\pi_n\). Since this map is also surjective, we conclude that \(\pi_n(Y)\) has to vanish. \(\square\)
For the proof of the next theorem, recall that any map \( f: U \to V \) can be factored as \( f = p \circ \phi \),
\[
f: U \xrightarrow{\phi} P(f) \xrightarrow{\psi} V,
\]
where \( p \) is a Serre fibration and \( \phi \) is a homotopy equivalence (‘mapping fibration’, see Section 5). We say that (up to homotopy), any map ‘can be turned into a fibration’.

**Theorem 11.3** (Postnikov tower). For any connected space \( X \), there is a ‘tower’ of fibrations
\[
P_1(X) \xleftarrow{\psi_1} P_2(X) \xleftarrow{\psi_2} P_3(X) \cdots
\]
and compatible maps \( f_i: X \to P_i(X) \) (compatible in the sense that \( \psi_n \circ f_{n+1} = f_n: X \to P_n(X) \)),
with the following properties:
(i) \( \pi_k(P_n(X)) = 0 \) for \( k > n \).
(ii) \( \pi_k(X) \to \pi_k(P_n(X)) \) is an isomorphism for \( k \leq n \) (and hence so is \( \pi_kP_n(X) \to \pi_kP_{n-1}(X) \)
for \( k < n \).
(iii) The fiber \( F_n \) of \( \psi_{n-1} \) has the property that \( \pi_n(F_n) \cong \pi_n(X) \) and \( \pi_k(F_n) = 0 \) for all \( k \neq n \).

**Remark 11.4.** A space like this fiber \( F_n \) with only one non-trivial homotopy group is called an *Eilenberg-MacLane space*. If \( Z \) is such a space with \( \pi_k(Z) = 0 \) for all \( k \neq n \) and \( \pi_n(Z) \cong A \), one says that \( Z \) is a \( K(A,n) \)-space (strictly speaking one always means the space \( Z \) together with a chosen isomorphism \( \pi_n(Z) \cong A \)). We will discuss these spaces in more detail in a later lecture.

With this terminology the situation of the theorem can be depicted as follows

\[
\begin{align*}
& \vdots \\
& P_3(X) \xleftarrow{f_3} F_3 = K(\pi_3(X), 3) \\
& \downarrow \psi_2 \downarrow \cdots \\
& P_2(X) \xleftarrow{f_2} F_2 = K(\pi_2(X), 2) \\
& \downarrow \psi_1 \downarrow \cdots \\
& X \xleftarrow{f_1} P_1(X)
\end{align*}
\]

where we used \( \xrightarrow{\text{fibration}} \) to denote a fibration.

**Proof of Theorem 11.3.** Let \( i_n: X \to Y_n \) be a space obtained from \( X \) by killing \( \pi_k(X) \) for all \( k > n \), i.e., such that
(i) \( (i_n)_*: \pi_k(X) \to \pi_k(Y_n) \) is an isomorphism for all \( k \leq n \).
(ii) \( \pi_k(Y_n) = 0 \) for all \( k > n \).

Such a space \( Y_n \) can be obtained by repeated application of the procedure of Lemma 11.2,
\[
X \subseteq Y_n^{(n+1)} \subseteq Y_n^{(n+2)} \subseteq \cdots
\]
where \( Y_n^{(n+1)} \) kills \( \pi_{n+1}(X) \) by adjoining \( (n+2) \)-cells, \( Y_n^{(n+2)} \) kills \( \pi_{n+2}(Y_n^{(n+1)}) \) by adjoining \( (n+3) \)-cells to \( Y_n^{(n+1)} \), and so on. The resulting space \( Y_n = \bigcup_{m>n} Y_n^{(m)} \), the union endowed with the weak topology, has the desired property, as is immediate from the fact that any map \( K \to Y_n \)
with $K$ compact (e.g., $K = S^k$ or $K = S^k \times [0,1]$) must factor through some $Y^{(m)}_n$. If you see what this construction does, then it is clear that there is a canonical inclusion $\phi_n: Y_{n+1} \to Y_n$ making the following diagram commute (we need to adjoin ‘more cells’ for $Y_n$ than for $Y_{n+1}$):

$$
\begin{array}{cc}
Y_{n+1} & \\
\downarrow^{i_{n+1}} & \\
X & \phi_n \\
\downarrow^{i_n} & \\
Y_n.
\end{array}
$$

Thus, $X$ is ‘approximated’ by smaller and smaller relative CW complexes

$$X \subseteq \ldots \subseteq Y_{n+1} \subseteq Y_n \subseteq \ldots \subseteq Y_2 \subseteq Y_1.$$

Now let $P_1(X) = Y_1$, and let $f_1: X \to P_1(X)$ be $i_1: X \to P_1(X)$. Let $P_2(X)$ be the space fitting into a factorization of

$$Y_2 \xrightarrow{\phi_1} Y_1 \xrightarrow{id} P_1(X)$$

into a homotopy equivalence $j_2$ followed by a fibration $\psi_1$. Next factor $j_2\phi_2$ in a similar way as $\psi_2 j_3$, and so on, all fitting into a diagram

$$
\begin{array}{cc}
\vdots & \\
\vdots & \\
X \xrightarrow{i_n} Y_n \xrightarrow{j_n} P_n(X) & \\
\downarrow & \phi_n & \downarrow \psi_n \\
\xrightarrow{i_{n-1}} Y_{n-1} \xrightarrow{j_{n-1}} P_{n-1}(X) & \\
\downarrow & \phi_{n-1} & \downarrow \psi_{n-1} \\
\xrightarrow{i_{n-2}} Y_{n-2} \xrightarrow{j_{n-2}} P_{n-2}(X) & \\
\vdots & \phi_{n-2} & \vdots \psi_{n-2} \\
\vdots & \\
X \xrightarrow{i_1} Y_1 \xrightarrow{j_1} P_1(X).
\end{array}
$$

Write $f_n: X \to P_n(X)$ for the composition $j_n i_n$, and denote the fiber of $\psi_{n-1}: P_n(X) \to P_{n-1}(X)$ by $F_n \subseteq P_n(X)$.

Now let us look at the homotopy groups. By construction we have (i) and (ii) above, and hence the same is true for $P_n(X)$ instead of $Y_n$:

(i) $(f_n)_*: \pi_k(X) \to \pi_k(P_n(X))$ is an isomorphism for all $k \leq n$.

(ii) $\pi_k(P_n(X)) = 0$ for all $k > n$.

We can feed this information in the long exact sequence of the fibration $F_n \subseteq P_n(X)$ $\xrightarrow{\psi_{n-1}} P_{n-1}(X)$, a part of which looks like

$$
\cdots \to \pi_{k+1}(P_n) \to \pi_{k+1}(P_{n-1}) \to \pi_k(P_n) \to \pi_k(F_n) \to \pi_k(P_{n-1}) \to \pi_k(P_{n-2}) \to \cdots
$$

where for simplicity we write $P_n$ for $P_n(X)$, and omit all base points from the notation. So, we clearly have:

(i) For $k > n$, the group $\pi_k(F_n)$ lies between two zero groups, hence is itself the zero-group.
(ii) For \( k < n \), the group \( \pi_k(F_n) \) lies between a surjection and an isomorphism,
\[
\bullet \rightarrow \pi_k(F_n) \rightarrow \bullet \cong \bullet,
\]
hence is zero again.

(iii) For \( k = n \), the relevant part of the sequence looks like
\[
0 \rightarrow 0 \rightarrow \pi_n(F_n) \rightarrow \pi_n(P_n) \rightarrow 0
\]
whence \( \pi_n(F_n) \) is isomorphic to \( \pi_n(P_n) \cong \pi_n(X) \).

This tells us that \( F_n \) is a \( K(\pi_n(X), n) \)-space and hence proves the theorem. \( \square \)

**Remark 11.5.** Much more can be said about these Postnikov towers: under some conditions, the fibration \( P_n \rightarrow P_{n-1} \) is even a fiber bundle.

2. **The Whitehead tower**

The Postnikov tower builds up the homotopy groups of \( X \) (together with all relations between them, such as the action of \( \pi_1 \) on \( \pi_n \) 'from below', by constructing for each \( n \) a space with homotopy groups \( \pi_1, \ldots, \pi_n \) only. There is also a construction 'from above', called the Whitehead tower of \( X \), as described in the following theorem.

**Theorem 11.6 (Whitehead tower).** Let \( X \) be a connected space. There exists a tower
\[
X \leftarrow W_1(X) \leftarrow W_2(X) \leftarrow W_3(X) \leftarrow \cdots
\]
with the following properties:
(i) \( \pi_k(W_n(X)) = 0 \) for \( k \leq n \).
(ii) The map \( \pi_k(W_n(X)) \rightarrow \pi_k(X) \) is an isomorphism for all \( k > n \).
(iii) The map \( W_n(X) \rightarrow W_{n-1}(X) \) is a fibration whose fiber is a \( K(\pi_n(X), n-1) \)-space.

**Proof.** As in the proof of the Postnikov tower, \( X \) can be approximated by extensions
\[
X \subseteq \cdots \subseteq Y_{n+1} \subseteq Y_n \subseteq \cdots \subseteq Y_2 \subseteq Y_1,
\]
where \( \pi_k(Y_n) = 0 \) for \( k > n \) and \( \pi_k(X) \rightarrow \pi_k(Y_n) \) is an isomorphism for \( k \leq n \). For \( X \subseteq Y \), let \( \bar{W}_n(X) \) be the space of paths in \( Y_n \) from the base point to \( X \), as in the pullback
\[
\begin{array}{ccc}
\bar{W}_n(X) & \longrightarrow & Y_n^{[0,1]} \\
\downarrow & & \downarrow \\
X \cong 1 \times X & \xrightarrow{x_0 \times \iota_n} & Y_n \times Y_n.
\end{array}
\]

So \( \bar{W}_n(X) \rightarrow X \) is a fibration. (Remember we used this fibration to describe relative homotopy groups of the pair \( (Y_n, X) \) in the exercises to Section 4.) These spaces fit naturally into a sequence
\[
X \leftarrow \bar{W}_1(X) \leftarrow \bar{W}_2(X) \leftarrow \bar{W}_3(X) \leftarrow \cdots
\]
Now turn these inclusions into fibrations (by factoring into a homotopy equivalence followed by a fibration as before) to obtain a diagram

$$
\begin{array}{ccccc}
W_1(X) & \rightarrow & W_2(X) & \rightarrow & W_3(X) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \cdots \\
W_1(X) & \leftarrow & W_2(X) & \leftarrow & W_3(X) & \leftarrow & \cdots
\end{array}
$$

where the lower horizontal maps are all fibrations and the vertical ones are homotopy equivalences.

Now let us look at the homotopy groups: We know $\pi_k(W_nX) \cong \pi_k(W_nX)$, and there are two fibrations to play with, viz $W_n(X) \rightarrow X$ and $W_n(X) \rightarrow W_{n-1}(X)$. The fiber of the first one is the loop space $\Omega Y_n$ of $Y_n$, and the fiber of the second one will be denoted $G_n$. Then the long exact sequence of $W_n(X) \rightarrow X$ looks like

$$
\cdots \rightarrow \pi_k(\Omega Y_n) \rightarrow \pi_k(W_nX) \rightarrow \pi_k(X) \rightarrow \pi_{k-1}(\Omega Y_n) \rightarrow \cdots
$$

or equivalently

$$
\cdots \rightarrow \pi_{k+1}(Y_n) \rightarrow \pi_k(W_nX) \rightarrow \pi_k(X) \rightarrow \pi_k(Y_n) \rightarrow \cdots
$$

But $\pi_k(Y_n) = 0$ for $k > n$ and $\pi_k(X) \rightarrow \pi_k(Y_n)$ is an isomorphism for $k \leq n$, so

$$
\pi_k(W_n(X)) \cong \pi_k(X), \quad k > n, \quad \text{and} \quad \pi_k(W_n) = 0, \quad k \leq n,
$$

and hence the same is true for $W_n$ instead of $W_n$. Next, the long exact sequence associated to $W_n(X) \rightarrow W_{n-1}(X)$ looks like

$$
\cdots \rightarrow \pi_{k+1}(W_n) \rightarrow \pi_{k+1}(W_{n-1}) \rightarrow \pi_k(G_n) \rightarrow \pi_k(W_n) \rightarrow \pi_k(W_{n-1}) \rightarrow \cdots
$$

(where we write $W_n$ for $W_n(X)$, etc), and we notice:

(i) if $k > n$ then $\pi_k(G_n)$ is squeezed in between two isomorphisms, so $\pi_k(G_n) = 0$.

(ii) if $k \leq n-2$ then $\pi_k(G_n)$ sits between two zero groups hence is zero itself.

(iii) if $k = n$ we obtain $\pi_{n+1}(W_n) \rightarrow \pi_{n+1}(W_{n-1}) \rightarrow \pi_n(G_n) \rightarrow 0$ and the first map is an isomorphism so that $\pi_n(G_n) = 0$.

(iv) in the remaining case $k = n-1$ the sequence looks like $0 \rightarrow \pi_n(W_{n-1}) \rightarrow \pi_{n-1}(G_n) \rightarrow 0$, so that we have an isomorphism $\pi_k(X) \cong \pi_n(W_{n-1}) \cong \pi_{n-1}(G_n)$.

Thus, this tells us that $G_n$ is a $K(\pi_n(X), n-1)$-space.

Note that the spaces $W_n(X)$ used in the proof of the Whitehead tower are precisely the homotopy fibers of the maps $i_n: X \rightarrow Y_n$ constructed in the proof of the Postnikov tower. The remaining work in the proof of Theorem 11.6 then consists of turning a certain sequence of maps between the homotopy fibers in a sequence of fibrations and analyzing what happens at the level of homotopy groups. This observation is sometimes referred to by saying that the Whitehead tower is obtained from the Postnikov tower ‘by passing to homotopy fibers’.

3. Examples of Eilenberg–Mac Lane spaces

In the construction of the Postnikov and Whitehead towers approximating a given space, $K(\pi, n)$-spaces naturally came up. We will conclude this lecture by giving a few of the most elementary examples of $K(\pi, n)$-spaces.
Remark 11.7. Recall that a $K(\pi, n)$-space, or an Eilenberg-MacLane space of type $(\pi, n)$, is a space $(X, x_0)$ such that $\pi_i(X, x_0) \cong \ast$ for all $i \neq n$ together with an isomorphism

$$\pi_n(X, x_0) \cong \pi.$$ 

Here $\pi$ can be a pointed set if $n = 0$, a group is $n = 1$, or an abelian group if $n \geq 2$. It can be shown that for such a $\pi$, a $K(\pi, n)$-space always exists, and is unique up to homotopy, although we will not give the general construction in this lecture.

Example 11.8 (Examples of $K(\pi, n)$-spaces).

(i) The circle $S^1$ is a $K(\mathbb{Z}, 1)$-space. Indeed, it is a connected space with fundamental group $\mathbb{Z}$, and one way to see that the higher homotopy groups vanish is to consider the universal covering space $\mathbb{R} \to S^1$. This is a fiber bundle with discrete fiber $F$ and contractible total space, so the long exact sequence gives us isomorphisms $0 = \pi_i(F) \cong \pi_{i+1}(S^1)$ for $i > 0$.

(ii) The same argument applies to wedges of spheres. Consider for example the 'figure eight' $S^1 \vee S^1$. Its fundamental group is the free group on two generators $\mathbb{Z} \ast \mathbb{Z}$. The universal cover of $S^1 \vee S^1$ can be explicitly described in terms of the 'grid' in the plane,

$$G = (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subseteq \mathbb{R}^2.$$

The map $w: G \to S^1 \vee S^1$ can be described by wrapping each edge of length 1 in the grid around one of the circles (in a way respecting orientations): say the vertical edges to the left hand circle and the horizontal edges to the right hand one. The universal cover $E$ of $S^1 \vee S^1$ is the space of homotopy classes of paths in $G$ which start in the origin, and $E \to S^1 \vee S^1$ is the composition

$$E \xrightarrow{\epsilon_1} G \xrightarrow{w} S^1 \vee S^1$$

(where $\epsilon_1$ is evaluation at the endpoint). The fiber of $\epsilon_1: E \to G$ over a given grid point $(n, m)$ with $n, m \in \mathbb{Z}$ is the set of 'combinatorial paths' from $(0, 0)$ to $(n, m)$: a sequence of alternating decisions: go left or go right, go up or go down, where successions of up-down and left-right cancel each other. Since each homotopy class of paths in $E$ has a unique such combinatorial description, the space $E$ is clearly contractible.

(iii) Recall that $\mathbb{RP}^n$, the real projective space of dimension $n$, is the space of lines in $\mathbb{R}^{n+1}$. It can be constructed as $S^n/\mathbb{Z}_2$ where the group $\mathbb{Z}_2 = \{0, 1\}$ acts by the antipodal map on the unit sphere

$$S^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \ldots + x_n^2 = 1 \}.$$

The embedding $S^n \to S^{n+1}$ sending $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, 0)$ sends $S^n$ to the 'equator' inside $S^{n+1}$, and is compatible with this antipodal action so that we get a commutative diagram

$$\begin{array}{cccc}
S^0 & \to & S^1 & \to & S^2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{RP}^0 & \to & \mathbb{RP}^1 & \to & \mathbb{RP}^2 & \to & \cdots 
\end{array}$$

There is a ‘natural’ CW decomposition of $S^{n+1}$, given inductively by a CW decomposition of $S^n$ with two $(n + 1)$-cells attached to it: the northern and the southern hemispheres. This makes $S^n$ into a CW complex with exactly two $k$-cells in each dimension $k \leq n$. One
can also take the union along the upper row of the diagram (with the weak topology) to obtain the *infinite-dimensional sphere*

\[ S^\infty = \bigcup_n S^n, \]

a CW complex with exactly two \( n \)-cells in each dimension \( n \). Note that since \( \pi_i(S^n) \cong 0 \) for \( i < k \) we also obtain \( \pi_i(S^\infty) \cong 0 \) for all \( i \geq 0 \). In other words, \( S^\infty \) is a weakly contractible CW complex, and hence by Whitehead’s theorem is contractible. In a similar way, we can take the union along the lower row in the above diagram to obtain

\[ \mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n, \]

a CW complex, the *infinite-dimensional real projective space*, with exactly one \( n \)-cell in each dimension \( n \). The long exact sequence of the covering projection \( S^n \to \mathbb{RP}^n \) with discrete fiber \( \mathbb{Z}_2 \) shows that

\[ \pi_i(\mathbb{RP}^n) = 0, \quad 1 < i < n, \quad \text{or} \quad i = 0, \quad \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2, \]

and by passing to the limit, one concludes that \( \mathbb{RP}^\infty \) is a \( K(\mathbb{Z}_2, 1) \)-space. (Alternatively, one can show that \( S^\infty \to \mathbb{RP}^\infty \) is still a covering projection with fiber \( \mathbb{Z}_2 \) to conclude that \( \mathbb{RP}^\infty \) is a \( K(\mathbb{Z}_2, 1) \).)

(iv) Recall that \( \mathbb{CP}^n \), the complex projective space of (complex) dimension \( n \), is the space of (complex) lines in \( \mathbb{C}^{n+1} \). It can be constructed as \( (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times \) where \( \mathbb{C}^\times = \mathbb{C} - \{0\} \) acts by multiplication; or, by choosing points on the line of norm 1, as the quotient of the unit sphere in \( \mathbb{C}^{n+1} \),

\[ \mathbb{CP}^n = S^{2n+1}/S^1, \]

where \( S^1 \subseteq \mathbb{C} \) again acts by multiplication. The quotient \( S^{2n+1} \to \mathbb{CP}^n \) has enough local sections (check this!), hence is a fiber bundle with fiber \( S^1 \). The embedding

\[ \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}: (z_0, \ldots, z_n) \mapsto (z_0, \ldots, z_n, 0) \]

induces maps

\[
\begin{array}{ccc}
S^{2n+1} & \to & S^{2n+3} \\
\downarrow & & \downarrow \\
\mathbb{CP}^n & \to & \mathbb{CP}^{n+1}
\end{array}
\]

and one can again take the union, to obtain a map \( S^\infty \to \mathbb{CP}^\infty \) with \( \mathbb{CP}^\infty \) the *infinite-dimensional complex projective space*. The space \( \mathbb{CP}^\infty \) is a quotient of \( S^\infty \) by \( S^1 \), and the map is again a fiber bundle. The spaces \( \mathbb{CP}^n \) have compatible CW complex structures, given by exactly one \( k \)-cell in each dimension \( k \leq n \). One way to see this is to represent a line in \( \mathbb{C}^{n+1} \) by a point

\[ z = (z_0, \ldots, z_n), \quad z_n \in \mathbb{R}, \quad z_n \geq 0, \quad \text{and} \quad ||z|| = z_0^2 + \ldots + z_n^2 = 1. \]

There is a unique way of doing this. Then the last coordinate \( t = z_n \) is uniquely determined by \( z' = (z_0, \ldots, z_{n-1}) \) (since \( t = \sqrt{1-||z'||} \)), and these \( (z_0, \ldots, z_{n-1}) \) form a disk of dimension \( 2n \). The boundary of this disk is given by \( ||z'|| = 1 \), in other words \( t = 0 \), and this is exactly the part already in \( \mathbb{CP}^{n-1} \). In any case, either of the two arguments at the end of the previous example shows that \( \mathbb{CP}^\infty \) is a \( K(\mathbb{Z}, 2) \)-space.
LECTURE 12: REPRESENTABLE FUNCTORS AND THE BROWN REPRESENTABILITY THEOREM

1. Representable functors

Let \( \mathcal{C} \) be a category. A functor \( F: \mathcal{C}^{\text{op}} \to \text{Sets} \) is called representable if there exists an object \( B = B_F \) in \( \mathcal{C} \) with the property that there is a natural isomorphism of functors

\[
\varphi: \mathcal{C}(-, B_F) \to F.
\]

Thus, for every object \( X \) in \( \mathcal{C} \), there is an isomorphism \( \varphi_X \) from the set of arrows \( \mathcal{C}(X, B_F) \) to the value \( F(X) \) of the functor. The naturality condition states that for any map \( f: Y \to X \) in \( \mathcal{C} \), the identity

\[
F(f)(\varphi_X(\alpha)) = \varphi_Y(\alpha \circ f)
\]

holds, for any \( \alpha: X \to B \). One usually expresses this in terms of a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(X, B) & \xrightarrow{\varphi_X} & F(X) \\
f^* & & f^* \\
\mathcal{C}(Y, B) & \xrightarrow{\varphi_Y} & F(Y),
\end{array}
\]

where \( f^* \) denotes the contravariant functoriality in \( f \); that is, \( f^* \) is composition with \( f \) on the left of the diagram, and \( f^* = F(f) \) on the right. By applying \( \varphi_B \) to the identity map \( B \to B \), which is generic in the sense that any element \( \xi \in F(X) \) can be obtained as \( \xi = f^*(\gamma) \), for a suitable \( f: X \to B \). Indeed, one can take \( f = \varphi_X^{-1}(\xi) \) and apply naturality to check that \( \xi = f^*(\gamma) \).

Clearly, if the functor \( F: \mathcal{C}^{\text{op}} \to \text{Sets} \) is representable, then it “respects” all colimits that exist in \( \mathcal{C} \). Since \( F \) is contravariant, these colimits are limits in \( \mathcal{C}^{\text{op}} \) and are turned into limits in \( \text{Sets} \) by \( F \). For example, a pushout diagram in \( \mathcal{C} \) as below on the left is turned into a pullback diagram on the right

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow f & & \downarrow k \\
B & \xrightarrow{h} & D
\end{array}
\quad
\begin{array}{ccc}
F(A) & \xleftarrow{g^*} & F(C) \\
\downarrow f^* & & \downarrow k^* \\
F(B) & \xleftarrow{h^*} & F(D),
\end{array}
\]

and a coproduct \( X = \coprod_{i \in I} X_i \) in \( \mathcal{C} \) is turned into a product \( F(X) = \prod_{i \in I} F(X_i) \).

Thus, for a functor \( F: \mathcal{C}^{\text{op}} \to \text{Sets} \) to be representable it is necessary that \( F \) turns colimits that exist in \( \mathcal{C} \) into limits in \( \text{Sets} \). It is not necessary to check this condition for all types of existing colimits, because some can be obtained from others. For example, coequalizers below on the left can be obtained from pushouts and (binary) coproducts, as indicated on the right:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \xrightarrow{g} C \\
\downarrow \vee & & \downarrow \vee \\
A & \xrightarrow{(f,g)} & B \coprod_{A} C
\end{array}
\]
where $\nabla$ denotes the “codiagonal”. Also, the colimit $X = \lim X_n$ of a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots$$

can be constructed from coequalizers and coproducts, as

$$\coprod X_n \xrightarrow{id} \coprod X_n \xrightarrow{f} \lim X_n,$$

(1.2)

where $f$ sends the summand $X_n$ to the summand $X_{n+1}$ via $f_n$.

Many algebraic structures can be expressed in terms of finite products and commutative diagrams, hence if $F: C^{\text{op}} \to \text{Sets}$ is representable, it sends such an algebraic structure to a similar structure in $\text{Sets}$. For example, a group $G$ in $C^{\text{op}}$, that is, a “cogroup” in $C$, given by comultiplication and counit

$$\ast \xleftarrow{\varepsilon} G \xrightarrow{\nabla} G \coprod G,$$

(* denotes the initial object in $C$) is turned into a group $F(G)$ with multiplication

$$\nabla^*: F(G) \times F(G) \cong F(G \coprod G) \to F(G)$$

and unit $\varepsilon^*$. Such coalgebraic structures are quite familiar in topology. As a basic example, recall that the group structure on $\pi_n(X,x_0) = [(S^n, \ast), (X,x_0)]$ comes from a cogroup structure on the sphere

$$\ast \xleftarrow{\nabla} S^n \xrightarrow{\nabla} S^n \vee S^n$$

given by the “pinch map” $\nabla$

(1.3)

So, if $F$ is a contravariant functor from the homotopy category of pointed spaces $\text{Ho}(\text{Top}_*)$ to $\text{Sets}$, then $F(S^n, \ast)$ is a group for each $n \geq 1$ (abelian for $n \geq 2$).

In $\text{Ho}(\text{Top}_*)$ and other cases we wish to study, the category $C$ is a pointed category: it has an object, usually denoted by $\ast$ or $\text{pt}$, which is both initial and terminal. So for any two objects $X$ and $Y$ there is a canonical arrow $X \to \ast \to Y$, and any representable functor $C^{\text{op}} \to \text{Sets}$, from a pointed category naturally takes values in the category $\text{Sets}$, of pointed sets.

Moreover, as we will see in our example, $C$ will have coproducts, but not very many other types of colimits. Instead, $C$ will have some “weak” colimits though: a weak colimit $A$ of a diagram $\{A_i\}_{i \in I}$ has the existence property of a colimit, but not the uniqueness property. In other words, for a compatible system of maps $\{A_i \to X\}_{i \in I}$ (a “cocone”) there is some $A \to X$ making the appropriate diagram commute, but it need not be unique. For example, if a square

$$\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow f & & \downarrow k \\
B & \xrightarrow{h} & D
\end{array}$$

is a weak pushout, then for any $u: B \to X$ and $v: C \to X$ with $u \circ f = v \circ g$, there is at least one $w: D \to X$ with $w \circ h = u$ and $w \circ k = v$, but there can be more such $w$. 
Note that, exactly as for ordinary colimits, one can construct weak coequalizers and weak colimits of sequences from weak pushouts and coproducts (cf. (1.1) and (1.2)). Also note that a representable functor necessarily sends weak colimits in \( \mathcal{C} \) to weak limits in \( \text{Sets} \) (or in the category \( \text{Sets}_* \) of pointed sets, if \( \mathcal{C} \) is pointed). Of course, weak limits are defined exactly like weak colimits, by dropping the uniqueness condition in the definition of ordinary limit.

The category \( \mathcal{C} \) that we are primarily interested in is the category \( \text{Ho}(\text{Top}_*) \) of pointed spaces and homotopy classes of pointed maps. This category has coproducts, the coproduct of a family \( \{X_i\}_{i \in I} \) being their wedge product \( \bigvee_{i \in I} X_i \), obtained from the disjoint union by identifying all the base points. The wedge product of the empty family also exists, and is the zero object, that is, a single point. However, most other types of colimits do not exist. On the other hand, if

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

is a pushout of pointed spaces (a pushout in the category \( \text{Top}_* \) and \( A \rightarrow X \) is a cofibration, then the homotopy extension property for cofibrations (see Lecture 8) at least says that this square is a weak pushout in \( \text{Ho}(\text{Top}_*) \). Thus, in \( \text{Ho}(\text{Top}_*) \), weak pushouts along cofibrations exist. From this fact, we can deduce the following:

**Proposition 12.1.** Let \( F \) be a contravariant functor from \( \text{Ho}(\text{Top}_*) \) into the category \( \text{Sets}_* \) of pointed sets. Suppose that \( F \) maps coproducts to products and pushouts along cofibrations in \( \text{Top}_* \) to weak pullbacks. Then

(i) If \( A \Rightarrow B \rightarrow C \) is a coequalizer in \( \text{Top}_* \) and the map \( A \vee A \rightarrow B \) is a cofibration, then \( FC \Rightarrow FB \Rightarrow FA \) is a weak coequalizer of pointed sets.

(ii) If \( X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \) is a sequence of cofibrations with colimit \( X \) in \( \text{Top}_* \), then \( F(X) \) is a weak limit of the inverse sequence \( F(X_0) \leftarrow F(X_1) \leftarrow F(X_2) \leftarrow \cdots \), that is, the map \( F(X) \rightarrow \varprojlim F(X_n) \) is a surjection of pointed sets.

**Proof.** Part (i) is clear from the description of coequalizers in terms of pushouts and coproducts as in diagram (1.1).

For part (ii), we need to do a bit more work. Recall that we can construct \( \varprojlim X_n \) as the coequalizer of the two maps

\[
\bigvee X_n \xrightarrow{i} \bigvee X_n,
\]

where the wedge is the coproduct in the category of pointed spaces, and where \( i \) is the identity while \( f \) sends the summand \( X_n \) to \( X_{n+1} \) by the given cofibration \( f_n \). Thus, this colimit is the pushout in the diagram

\[
\begin{array}{ccc}
(\bigvee X_n) \vee (\bigvee X_n) & \xrightarrow{(i,f)} & \bigvee X_n \\
\downarrow & & \downarrow \quad q \\
\bigvee X_n & \xrightarrow{q} & \varprojlim X_n
\end{array}
\]

as discussed before. We have written \( q \) for the canonical map \( q: \bigvee X_n \rightarrow \varprojlim X_n \), with components \( q_n: X_n \rightarrow \varprojlim X_n \). The problem is that \( (i,f) \) is not (necessarily) a cofibration. To resolve this, we are going to “thicken” the colimit and construct a “telescope” \( T \), which decomposes the
pushout (1.4) above into a composition of two pushouts:

$$
\begin{align*}
\bigvee X_n \vee \bigvee X_n \xrightarrow{(i', f')} \bigvee(X_n \land I^+) 
\xrightarrow{pr_1} \bigvee X_n
\end{align*}
$$

To see what these maps are, write points of \( \bigvee X_n \) as pairs \((n, x)\), where \( x \in X_n \), and points of \( \bigvee(X \land I^+) \) as triples \((n, x, t)\), where \( x \in X_n \) and \( t \in I \). Then

\[
i'(n, x) = (n, x, 1) \quad \text{and} \quad f'(n, x) = (n + 1, f_n(x), 0).
\]

So points of \( T \) are equivalence classes of triples \((n, x, t)\), with identifications \((n, x_0, t) \sim (n, x_0, t')\) for the base point \( x_0 \) coming from the definition of \( X \land I^+ \), and identifications \((n, x, 1) \sim (n + 1, f_n(x), 0)\) coming from the definition of the pushout. Let us write \([n, x, t] = p(n, x, t)\) for the equivalence class. Then \( v: \bigvee X_n \to T \) is the map given on a summand \( X_n \) by

\[
v_n: X_n \to T, \quad v_n(x) = [n, x, 1]
\]

and \( \pi: T \to \lim X_n \) is the obvious projection, \( \pi[n, x, t] = q_n(x) \).

Now observe that \((i', f')\) is a cofibration. Indeed, it is a wedge (coproduct) of cofibrations

\[
i_0': X_0 \to X_0 \land I^+ \quad \text{(sending } x \mapsto (x, 1))
\]

and for \( n \geq 0 \)

\[
X_n \land X_{n+1} \xrightarrow{f_n \vee \text{id}} X_{n+1} \land X_{n+1} \to X_{n+1} \land I^+,
\]

where the second map is the standard cofibration mapping the two copies of \( X_{n+1} \) to \( X_{n+1} \times \{0\} \) and \( X_{n+1} \times \{1\} \), respectively.

We are now ready to prove that the map \( F(\lim X_n) \to \lim F(X_n) \) is surjective. Choose a sequence \( \xi_n \in F(X_n) \) with \( (f_n)^*\xi_{n+1} = \xi_n \) \((n \geq 0)\). These \( \xi_n \) together make up an element \( \xi \) in \( \prod F(X_n) \cong F(\bigvee X_n) \). Let \( \xi_n \in F(X \land I^+) \) be obtained from \( \xi_n \) by applying \( F \) to the projection \( \text{pr}_1: X_n \land I^+ \to X_n \). Then \( \xi_n \) together define an element \( \xi \in F(\bigvee(X_n \land I^+)) \).

The assumption that \( (f_n)^*\xi_{n+1} = \xi_n \) means precisely that \( (i')^*\xi = f'(\xi) = \xi \). So by applying (i) to the pushout square on the left of (1.5) we find a \( \zeta \in F(T) \) with \( v^*\zeta = \xi \) and \( p^*\zeta = \xi \).

In particular, \( (v_n)^*\zeta = \xi_n \).

We now wish to “push down” \( \zeta \) to an element \( \eta \in F(\lim X_n) \). To this end, we construct a map \( w: \lim X_n \to T \). Consider the maps \( v_n: X_n \to T \), and observe that each triangle

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & X_{n+1} \\
\downarrow{v_n} & & \downarrow{v_{n+1}} \\
T & \xrightarrow{} & T \\
\end{array}
\]

commutes up to homotopy. Indeed, \( v_n(x) = [n, x, 1] = [n + 1, f_n(x), 0] \) and \( v_{n+1}(f_n(x)) = [n + 1, f_n(x), 1] \), which are connected by the homotopy sending \( x \) to \([n + 1, f_n(x), t]\) for \( 0 \leq t \leq 1 \).

We can now successively apply the homotopy extension property to the cofibrations \( f_0, f_1, f_2, \ldots \) and replace the \( v_i \) by homotopic maps \( w_i \simeq v_i \) so that \( w_{n+1} \circ f_n = w_n \). This gives a map

\[
w: \lim X_n \to T \quad \text{with} \quad w \circ q_n = w_n \simeq v_n.
\]

Let \( \eta = w^*\zeta \). Then \( \eta \) is the element in \( F(\lim X_n) \) we are looking for, because

\[
(q_n)^*\eta = (q_n)^*w^*\zeta = (w \circ q_n)^*\zeta = (w_n)^*\zeta = (v_n)^*\zeta = \xi_n,
\]
proving that \( F(X) \to \lim_{\leftarrow} F(X_n) \) is a surjection. \( \Box \)

**Exercise 12.2.** Prove the result in the last part of the proof stating that, by applying the homotopy extension property to the cofibrations \( f_i \), we can replace the maps \( v_i \) by homotopic maps \( w_i \) so that \( w_{n+1} \circ f_n = w_n \).

The kinds of colimits mentioned in the proposition have a special property, viz. they are “homotopy invariant”. For coproducts this is clear: a family of pointed homotopy equivalences \( X_i \xrightarrow{\simeq} Y_i \) (where \( i \) ranges over some index set \( I \)) induces a pointed homotopy equivalence \( \bigvee X_i \xrightarrow{\simeq} \bigvee Y_i \). For pushouts along cofibrations, it is a bit more complicated. The next statement can be proved using the properties of cofibrations stated at the end of Lecture 8.

**Proposition 12.3.** Let \( F: \text{Ho}((\text{Top}_*)^{\text{op}}) \to \text{Sets}_* \) be a contravariant functor, from the homotopy category of pointed spaces to pointed sets. Suppose that \( F \) sends each pushout

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \cup_A C
\end{array}
\]

of two cofibrations to a weak pullback in \( \text{Sets}_* \). Then \( F \) sends each pushout along a cofibration to a weak pullback.

**Proof.** If we have a pushout diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \cup_A X
\end{array}
\]

of a map \( f: A \to X \) along a cofibration \( A \to B \), we can factor \( A \to X \) as a cofibration followed by a homotopy equivalence, and construct the pushout in two steps:

\[
\begin{array}{ccc}
A & \longrightarrow & M_f \xrightarrow{\simeq} X \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \cup_A M_f \xrightarrow{\simeq} B \cup_A X.
\end{array}
\]

Then the lower map on the right is a homotopy equivalence by Corollary 8.15 from Lecture 8. If we apply \( F \) to this diagram, we obtain a diagram

\[
\begin{array}{ccc}
F(A) & \longleftarrow & F(M_f) \xrightarrow{\simeq} F(X) \\
\uparrow & & \uparrow \\
F(B) & \longleftarrow & F(B \cup_A M_f) \xrightarrow{\simeq} F(B \cup_A X)
\end{array}
\]

in which the square on the left is a weak pullback by hypothesis, while in the one on the right, the horizontal maps are isomorphisms. It follows that the large rectangle is also a weak pullback. \( \Box \)

2. **Brown representability theorem**

Let us summarize the discussion so far. Suppose that

\[ F: \text{Ho}(\text{Top}_*)^{\text{op}} \to \text{Sets}_* \]

is a functor having the following two properties:
Then we also have

(i) \( F(\bigvee_{i \in I} X_i) \to \prod_{i \in I} F(X_i) \) is an isomorphism, for any family of pointed spaces \( \{X_i\}_{i \in I} \).

(ii) \( F(B \cup_A C) \to F(B) \times_{F(A)} F(C) \) is a surjection for any two cofibrations \( A \to B \) and \( A \to C \).

Then we also have

(iii) \( F(*) = * \)

(iv) If we have a pushout of pointed spaces

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

in which \( i \) is a cofibration, then \( F(Y) \to F(B) \times_{F(A)} F(X) \) is a surjection.

(v) If \( A \implies B \to C \) is a coequalizer of pointed spaces in which the two maps form a cofibration \( A \vee A \to B \), then \( F(C) \to F(B) \) maps surjectively to the equalizer of \( F(B) \implies F(A) \).

(vi) If \( X_0 \to X_1 \to X_2 \to \cdots \) is a sequence of cofibrations, then \( F(X) \to \lim F(X_n) \) is a surjection.

Our aim is now to show that conditions (i) and (ii) are in fact enough to show that \( F \) is representable, at least when we restrict ourselves to connected CW-complexes.

**Theorem 12.4** (Brown representability theorem). Let \( F \) be a contravariant functor from the homotopy category of pointed connected CW-complexes to pointed sets. If \( F \) satisfies conditions (i) and (ii) above (for any pointed connected CW-complexes \( X, A, B, C \)), then \( F \) is representable.

**Remark 12.5.**

(i) Recall that this means that there is a space \( B = B_F \) (itself a pointed CW-complex) for which there is a natural isomorphism

\[ \varphi_X : [X, B_F] \to F(X), \]

for any pointed connected CW-complex \( X \). This space \( B_F \) is called a classifying space for \( F \). Recall also that when such a \( \varphi \) exists, it is completely determined by a generic element \( \gamma \in F(B_F) \).

(ii) Suppose that \( (B_1, \gamma_1) \) and \( (B_2, \gamma_2) \) are two classifying spaces for \( F \), with generic elements \( \gamma_1 \) and \( \gamma_2 \), respectively. Then there exists a homotopy equivalence \( f : B_1 \to B_2 \) with \( f^*(\gamma_2) = \gamma_1 \). In other words, the pair \( (B, \gamma) \) of a classifying space and its generic element is unique up to homotopy. Indeed, writing \( \mathcal{C} \) for the category of pointed connected CW-complexes and homotopy classes of maps, there are natural isomorphisms

\[ \varphi_X^1 : \mathcal{C}(X, B_1) \to F(X) \text{ and } \varphi_X^2 : \mathcal{C}(X, B_2) \to F(X) \]

defined by \( \varphi_X^i(f) = f^*(\gamma_i) \), for \( i = 1, 2 \). Then

\[ (\varphi_{B_2}^1)^{-1}(\gamma_1) : B_2 \to B_1 \text{ and } (\varphi_{B_1}^2)^{-1}(\gamma_2) : B_1 \to B_2 \]

are mutually inverse maps in the category \( \mathcal{C} \).

(iii) We can use the Whitehead theorem to make a slightly different statement. Let us say that \( (B, \gamma) \) is a spherical classifying space for \( F \) if \( \gamma \in F(B) \) is an element inducing an isomorphism

\[ \gamma_* : [S^n, B] \to F(S^n), \quad \gamma_*(f) = f^*(\gamma) \]
for any sphere $S^n$ ($n > 0$; $S^0$ is not connected). If $(B_1, \gamma_1)$ and $(B_2, \gamma_2)$ are two such spherical classifying spaces in the category $\mathcal{C}$, and $f: B_1 \to B_2$ is a map with $f^*(\gamma_2) = \gamma_1$, then $f$ induces isomorphisms

$$\pi_n(B_1) \to \pi_n(B_2)$$

between the homotopy groups for each $n > 0$ (we dropped the basepoints of $B_1$ and $B_2$ from the notation). Since $B_1$ and $B_2$ are pointed connected CW-complexes, this means that $f$ is a weak homotopy equivalence, hence a homotopy equivalence by the Whitehead theorem.

Let us now turn to the proof of Brown representability theorem. It is based on the following lemmas.

**Lemma 12.6.** Let $X$ be a pointed CW-complex and let $\xi \in F(X)$. Then there exists a spherical classifying space $(B, \gamma)$ for $F$ with a cofibration $f: X \to B$ with $f^*(\gamma) = \xi$.

**Lemma 12.7.** Any spherical classifying space $(B, \gamma)$ for $F$ is a classifying space. (Thus, $\gamma_*: [X, B]_* \to F(X)$ is an isomorphism for any pointed connected CW-complex $X$, not just for spheres.)

Indeed, Brown’s theorem follows by taking $X$ to be a point in Lemma 12.6, and then applying Lemma 12.7 to the spherical classifying space provided by Lemma 12.6. We will now first show that Lemma 12.7 follows from Lemma 12.6, and then prove Lemma 12.6.

**Proof of Lemma 12.7 (using Lemma 12.6).** Let $X$ be a pointed connected CW-complex, and let $(B, \gamma)$ be a spherical classifying space for $F$.

We first prove that $\gamma_*: [X, B]_* \to F(X)$ is a surjection. Let $\xi \in F(X)$. Form the wedge

$$X \xrightarrow{i} X \vee B \xrightarrow{j} B.$$ 

Since $F(X \vee B) \cong F(X) \times F(B)$ (by an isomorphism identifying $i^*$ and $j^*$ with the projections), we find an element $(\xi, \gamma) \in F(X \vee B)$ with $i^*(\xi, \gamma) = (\xi, \gamma)$ and $j^*(\xi, \gamma) = \gamma$. By Lemma 12.6, there is a spherical classifying space $(B', \gamma')$ and a cofibration

$$f: X \vee B \to B'$$

with $f^*(\gamma') = (\xi, \gamma)$. Thus $(f \circ i)^*(\gamma') = \xi$ and $(f \circ j)^*(\gamma') = \gamma$. But then $f \circ j: B \to B'$ is a homotopy equivalence by Remark 12.5(iii). If $g: B' \to B$ is a homotopy inverse for $f \circ j$, then $g \circ f \circ i: X \to B$ is a map with $(g \circ f \circ i)^*(\gamma) = \xi$

$$\begin{array}{c}
X \xrightarrow{i} X \vee B \xrightarrow{j} B \\
f \downarrow \approx \downarrow g \\
\quad \quad \quad \quad \quad \quad \quad B'.
\end{array}$$

This proves that $\gamma_*: [X, B]_* \to F(X)$ is a surjection.

Next, we prove that $\gamma_*: [X, B]_* \to F(X)$ is injective. Suppose that $f$ and $g$ are two maps $X \Rightarrow B$ with $f^*(\gamma) = g^*(\gamma) \in F(X)$. Consider the diagram

$$\begin{array}{c}
X \vee X \xrightarrow{f \vee g} B \\
\downarrow \triangleright \quad \downarrow \quad \downarrow h \\
X \xrightarrow{\varepsilon} X \wedge I^+.
\end{array}$$
where $X \wedge I^+$ is the reduced cylinder $(X \times I)/\{x_0\} \times I$. We wish to find a map $h: X \wedge I^+ \to B$ with $h \circ i = f \vee g$ because this would show that $f \simeq g$. To this end, form the pushout

$$
\begin{array}{ccc}
X \vee X & \xrightarrow{f \vee g} & B \\
\downarrow i & & \downarrow u \\
X \wedge I^+ & \xrightarrow{v} & W.
\end{array}
$$

Now $F(X \vee X) = F(X) \times F(X)$ and $(f \vee g)^*(\gamma) = (f^*(\gamma), g^*(\gamma))$ under this identification. Let $\zeta = e^* \circ f^* \circ \gamma = e^* \circ g^* \circ \gamma \in F(X \wedge I^+)$. Then $i^*(\zeta) = (f \vee g)^*(\gamma)$, so since $F$ transforms the pushout into a weak pullback, there exists an $\eta \in F(W)$ with $v^*(\eta) = \zeta$ and $u^*(\eta) = \gamma$. By Lemma 12.6, there exists a spherical classifying space $(B', \gamma')$ and a cofibration $w: W \to B'$ with $w^*(\gamma') = \eta$. Then $(w \circ u)^*(\gamma') = \gamma$, so $w: B \to B'$ is a homotopy equivalence.

In particular, there is a map $p: B' \to B$ with $p^*(\gamma) = \gamma'$ and $p \circ w \circ v \simeq id$. Then, $p \circ w \circ v \circ i = p \circ w \circ u(f \vee g) \simeq f \vee g$, so by the homotopy extension property applied to the cofibration $X \vee X \to X \wedge I^+$ we find a map $q: X \wedge I^+ \to B$ with $q \simeq p \circ w \circ v$ and $q \circ i = f \vee g$. In particular, $q$ is a homotopy between $f$ and $g$.

**Proof of Lemma 12.6.** Let $X$ be a pointed connected CW-complex and $\xi \in F(X)$. We are going to construct a sequence of cofibrations

$$X \subseteq B^1 \subseteq B^2 \subseteq B^3 \subseteq \cdots$$

together with elements $\gamma^n \in F(B^n)$ (for $n > 0$), such that the map

$$(\gamma^n)_*: [S^n, B^n]_* \to F(S^n)$$

which sends $f$ to $f^*(\gamma^n)$, is a surjection for $q = n$ and a bijection for $0 < q < n$. Moreover, the $\gamma^i$ will be compatible with each other and with $\xi$ in the obvious sense that the image of $X \to B^n \to B^{n+1}$ under $F$ sends $\gamma^{n+1}$ to $\gamma^n$ and then to $\xi$. These $B^n$ will be constructed in quite a straightforward way, by attaching cells, much as in the proof of the CW-approximation theorem. For $n = 1$, let

$$B^1 = X \vee \bigvee_\xi S^1,$$

where $\zeta$ ranges over all elements of $F(S^1)$ and $S^1$ is a copy of $S^1$. Then, by property (i) on page 6, $F(B^1) \cong F(X) \times \prod_\xi F(S^1_\xi)$, and we let $\gamma^1_\xi$ be the element with coordinate $\xi$ on $F(X)$ and coordinate $\gamma^1$ on the factor $F(S^1_\xi)$. Then, for the inclusion $i_\xi: S^1_\xi \to B^1$ we have $i_\xi^*(\gamma^1) = \zeta$.

In particular, $[S^1, B^1]_* \to F(S^1)$ is surjective.

Suppose that $(B^n, \gamma^n)$ has been constructed with the desired properties. In particular, $(\gamma^n)_*: [S^n, B^n]_* \to F(S^n)$ is a surjection of pointed sets. In fact, it is a surjection of groups, because $S^n$ is an $H$-cogroup, cf. (1.3). Let $K$ be the kernel of $(\gamma^n)_*$. Let $B^{n+\frac{1}{2}}$ be the space obtained form $B^n$ by attaching an $(n+1)$-cell along the attaching map $k: S^n \to B^n$, one $k$ for each homotopy class $[k]$ in this kernel $K$. Thus, we have a pushout:

$$
\begin{array}{ccc}
P_k S^n & \xrightarrow{\bigvee_k S^n} & B^n \\
\downarrow \cup k & & \downarrow \\
P_k e^{n+1} & \xrightarrow{\bigvee_k e^{n+1}} & B^{n+\frac{1}{2}}.
\end{array}
$$

Since $e^{n+1}$ is contractible, $F(e^{n+1})$ is a point, so the pullback of $F(B^n) \to \prod_k F(S^n)$ along $\prod F(e^{n+1}) \to \prod F(S^n)$ is the kernel of the map $F(B^n) \to \prod_k F(S^n)$, sending $\gamma^n$ to the element
with coordinate $k^*(\gamma^n) = (\gamma^n)_*(k)$ on the factor $k$. The map $F(B^{n+\frac{1}{2}}) \to F(B^n)$ surjects onto this kernel (by property (iv) on page 6), so there is an element $\gamma^{n+\frac{1}{2}} \in B^{n+\frac{1}{2}}$ with $j^*(\gamma^{n+\frac{1}{2}}) = \gamma^n$.

For each $q \leq n$ we now have a diagram

$$
\begin{array}{ccc}
[S^q, B^{n+\frac{1}{2}}] & \xrightarrow{(\gamma^{n+\frac{1}{2}})_*} & F(S^q) \\
\downarrow j_* & & \downarrow (\gamma)_* \\
[S^q, B^n] & \xrightarrow{(\gamma)_*} & F(S^q)
\end{array}
$$

By cellular approximation, $j_*$ is an isomorphism for $q < n$, and hence $(\gamma^{n+\frac{1}{2}})_*$ is because $(\gamma^n)_*$ is by induction hypothesis. Moreover, $(\gamma^{n+\frac{1}{2}})_*$ is surjective for $q = n$ because $(\gamma^n)_*$ is. It is also a surjection for $q = n$, because if $k: S^n \to B^{n+\frac{1}{2}}$ is (or represents) a homotopy class with $(\gamma^{n+\frac{1}{2}})_*(k) = 0$, then by cellular approximation $k$ is homotopic to $j \circ k'$ for a map $k': S^n \to B^n$, and $(\gamma^n)_*(k') = 0$ so $k'$ (or more precisely its homotopy class) lies in $K$. Then $j_*(k') = k = 0$ in $[S^n, B^{n+\frac{1}{2}}]_*$ by construction of $B^{n+\frac{1}{2}}$.

Finally, we construct $B^{n+1}$ from $B^{n+\frac{1}{2}}$ much as we constructed $B^1$ from $X$, as

$$
B^{n+1} = B^{n+\frac{1}{2}} \vee \bigvee_{\zeta} S^{n+1}_\zeta,
$$

where $\zeta$ ranges over all elements of $F(S^{n+1})$ and each $S^{n+1}_\zeta$ is a copy of $S^{n+1}$. Then

$$
F(B^{n+1}) \cong F(B^{n+\frac{1}{2}}) \times \prod_{\zeta} F(S^{n+1}_\zeta)
$$

has a canonical element $\gamma^{n+1}$ with coordinates $\gamma^{n+\frac{1}{2}}$ and $\zeta$. Moreover, we have that the map

$$
(\gamma^{n+1})_*: [S^q, B^{n+1}]_* \to F(S^q)
$$

is an isomorphism for $q \leq n$ (as before, for $B^{n+\frac{1}{2}}$) and a surjection for $q = n + 1$ (by construction).

To conclude the proof, let $B = \varinjlim B^n$ and use property (vi) on page 6 to find an element $\gamma \in F(B)$ such that for every $n$, the element $\gamma$ is mapped to $\gamma^n$ by $F(B^n \to B)$. Then $(B, \gamma)$ is a spherical classifying space.

This completes the proof of the Brown representability theorem. \[\square\]
The homotopy excision theorem is a result about relative homotopy groups, or homotopy groups of pairs. Recall that for a pair \((X, A)\), we defined \(\pi_i(X, A)\) as the set of homotopy classes of maps \(I^i \to X\) which send the top face \(I^{i-1} \times \{1\}\) to \(A\) and the rest of the boundary \(J^{i-1} = I^{i-1} \times \{0\} \cup \partial I^{i-1} \times I\) to the base point \(x_0\) (which we will consistently omit from the notation). In other words,

\[
\pi_i(X, A) = [(I^i, \partial I^i, J^{i-1}), (X, A, x_0)].
\]

This is a pointed set for \(i = 1\), a group for \(i = 2\), and an abelian group for \(i \geq 3\). Moreover, these groups fit into a long exact sequence

\[
\ldots \to \pi_i(A) \to \pi_i(X) \to \pi_i(X, A) \to \pi_{i-1}(A) \to \ldots
\]

We didn’t define \(\pi_0(X, A)\), but we can set \(\pi_0(X, A) = \text{cok}(\pi_0(A) \to \pi_0(X))\) so that the long exact sequence can be prolonged so as to end as \(\ldots \to \pi_0(A) \to \pi_0(X) \to \pi_0(X, A) \to 0\).

Recall that the pair \((X, A)\) is \(n\)-connected if \(\pi_i(X, A) = 0\), \(i \leq n\). By the long exact sequence, this is the same as asking that \(\pi_i(A) \to \pi_i(X)\) is an isomorphism for \(i < n\) and a surjection for \(i = n\) (in other words, that the inclusion \(A \to X\) is an \(n\)-equivalence). Recall further that the pair \((X, A)\) is called a relative CW complex if \(X\) is obtained from \(A\) by successively attaching cells.

The first part of the title refers to the following statement in which we consider a situation as depicted in the following diagram

\[
\begin{array}{ccc}
C & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & X.
\end{array}
\]

**Theorem 13.1. (Homotopy excision theorem)**

Let \(C\) be any space, and let \((A, C)\) and \((B, C)\) be relative CW complexes. Write \(X = A \cup_C B\) for the union of \(A\) and \(B\) (the pushout under \(C\)). If \((A, C)\) is \(m\)-connected and \((B, C)\) is \(n\)-connected, then

\[
\pi_i(A, C) \to \pi_i(X, B)
\]

is an isomorphism for \(i < m + n\), and a surjection for \(i = m + n\).

To put it differently, if \(C \to A\) is an \(m\)-equivalence and \(C \to B\) an \(n\)-equivalence, then the induced map of pairs \((A, C) \to (X, B)\) is an \((m + n)\)-equivalence. The proof of this theorem will occupy this lecture and part of the next. But before we go into the proof, we will mention the following important application. Recall the suspension functor \(\Sigma(X) = X \wedge S^1\) which can also be described by the following pushout

\[
\begin{array}{ccc}
X & \longrightarrow & C'X \\
\downarrow & & \downarrow \\
CX & \longrightarrow & \Sigma X
\end{array}
\]
where $CX$ and $C'X$ are two copies of the (reduced) cone of $X$. In the special case of $X = S^p$, we have $\Sigma(S^p) \cong S^{p+1}$ so that the suspension induces a suspension homomorphism (see also Lecture 4)

$$S: \pi_i(X) \to \pi_{i+1}(\Sigma X).$$

**Theorem 13.2. (Freudenthal suspension theorem)**

Let $(X, x_0)$ be an $(n-1)$-connected CW complex. Then the suspension homomorphism

$$S: \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i < 2n - 1$, and a surjection for $i = 2n - 1$.

**Proof.** Notice that the long exact sequence of the pair $(Y, A)$ gives an isomorphism $\pi_i(Y) \to \pi_i(Y, A)$ for any $i$ if $A$ is contractible, and an isomorphism $\partial: \pi_{i+1}(Y, A) \to \pi_i(A)$ if $Y$ is contractible. Now consider the square:

$$\begin{array}{ccc}
\pi_{i+1}(CX, X) & \cong & \pi_{i+1}(\Sigma X, C'X) \\
\downarrow \partial & & \downarrow \cong \\
\pi_i(X) & \longrightarrow & \pi_{i+1}(\Sigma X)
\end{array}$$

Since the two copies $CX$ and $C'X$ of the cone are contractible, we have the two vertical isomorphisms which are induced by the respective long exact sequences. The upper horizontal map is induced by the inclusion $(CX, X) \to (\Sigma X, C'X)$ while the bottom horizontal map can be identified with the suspension homomorphism, the map showing up in the statement of the theorem. We leave it as an exercise to verify that the diagram commutes. To prove the theorem, it thus suffices to check that the upper map is an isomorphism in an appropriate range of $i$'s. To this end, apply the excision theorem to $\Sigma(X) = CX \cup_X C'X$. Indeed, since $CX$ is contractible, the long exact sequence of the pair $(CX, X)$ shows that $(CX, X)$ is $n$-connected if $X$ is $(n-1)$-connected. So the upper horizontal map is an isomorphism for $i + 1 < 2n$, and a surjection for $i + 1 = 2n$, exactly as stated in the theorem. \[\square\]

**Example 13.3.** The $n$-sphere $S^n$ is a CW complex with one 0-cell and one $n$-cell $(n > 0)$, so is surely $(n-1)$-connected by the cellular approximation theorem. So by the Freudenthal suspension theorem,

$$S: \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n - 1$. In particular,

$$S: \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$$

is an isomorphism if $n < 2n - 1$, i.e. if $n \geq 2$. We already know that $\pi_1(S^1) \cong \mathbb{Z}$, while the long exact sequence of the Hopf fibration

$$S^1 \to S^3 \to S^2$$

together with the fact that $\pi_i(S^3) = 0$ for $i < 3$ readily shows that $\partial: \pi_2(S^2) \to \pi_1(S^1)$ is an isomorphism. Thus,

$$\pi_n(S^n) \cong \mathbb{Z}, \quad \text{for all } n \geq 1.$$

**Perspective 13.4.** For an arbitrary pointed CW complex $X$, the Freudenthal suspension theorem and the connectivity of $\Sigma X$ give that $\Sigma^n X$ is always $(n-1)$-connected. Thus, the map

$$S: \pi_i(\Sigma^n X) \to \pi_{i+1}(\Sigma^{n+1} X)$$
is an isomorphism for \( i < 2n - 1 \). This implies that for a fixed value of \( k \), the maps in the sequence

\[
\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \cdots \to \pi_{k+i}(\Sigma^i X) \to \pi_{k+i+1}(\Sigma^{i+1} X) \to \cdots
\]

eventually become isomorphisms. (More precisely, \( \pi_{k+i}(\Sigma^i X) \to \pi_{k+i+1}(\Sigma^{i+1} X) \) is certainly an isomorphism for \( k + i < 2i - 1 \), or \( k + 1 < i \).) The eventual value of this sequence is called the \( k \)-th **stable homotopy group** of \( X \), denoted

\[
\pi_k^s(X).
\]

These (abelian) stable homotopy groups are still extremely informative, while being more computable than the ordinary (‘unstable’) ones. In a sense, they sit between the unstable homotopy groups and the homology groups and form a central subject of study in algebraic topology.

More generally, given a sequence of pointed spaces \( X_0, X_1, X_2, \ldots \) related by structure maps \( \sigma_k: \Sigma X_k \to X_{k+1} \), one can form a sequence

\[
\pi_k(X_0) \to \pi_{k+1}(X_1) \to \pi_{k+2}(X_2) \to \cdots \to \pi_{k+i}(X_i) \to \pi_{k+i+1}(X_{i+1}) \to \cdots
\]

(by using the suspension homomorphisms together with the homomorphism induced by the structure maps). Such sequences of spaces, called **spectra**, are the main objects of ‘stable homotopy theory’. Any pointed space \( X \) gives rise to a spectrum by taking \( X_n = \Sigma^n X \), the **suspension spectrum** \( \Sigma^\infty X \) of \( X \). In a specific sense, the passage to the (homotopy) category of spectra is a good approximation of the (homotopy) category of spaces, which has more structure and is more tractable.

We will now turn to the proof of the excision theorem, and start with a few reductions to simpler cases. The first one is concerned with the dimension of the cells to be added to \( C \) in an \( m \)-connected pair \( (A, C) \). Notice that if \( A \) is obtained from \( C \) by attaching cells of dimension larger than \( m \) only, then the pair \( (A, C) \) is automatically \( m \)-connected (see an earlier lecture). In fact, the proof of the CW approximation theorem shows that the converse is also true as the following lemma shows.

**Lemma 13.5.** Let \( i: C \to A \) be an inclusion defining an \( n \)-connected pair of spaces \( (A, C) \). Then there is a relative CW complex \( i': C \to A' \) and a weak homotopy equivalence \( e: A' \to A \) for which \( e i' = i \), and where \( A' \) is obtained from \( C \) by attaching cells of dimension \( > n \) only.

**Proof.** We follow the same strategy as in the CW approximation theorem, and build up a CW complex by adding cells which represent elements of \( \pi_i(A) \) respectively kill elements which should not be there. Now all of \( \pi_i(A) \) for \( i < n \) is already represented by maps \( S^i \to C \) since \( \pi_i(C) \to \pi_i(A) \) for \( i < n \), so the first step in this process consists of killing the kernel of the surjection \( \pi_n(C) \to \pi_n(A) \) by attaching cells to \( A \). This gives an extension

\[
C \to \tilde{A}'_{n+1}
\]

by \( (n+1) \)-cells, together with a map \( \tilde{A}'_{n+1} \to A \) making the diagram

\[
\begin{array}{ccc}
C & \to & \tilde{A}'_{n+1} \\
\downarrow & & \downarrow \\
A & \to & \tilde{A}'_{n+1}
\end{array}
\]

commute. In the next step, we attach \( (n+1) \)-cells to the base point of \( \tilde{A}'_{n+1} \) to represent all elements (or a set of generators) of \( \pi_{n+1}(A) \), giving a space \( A'_{n+1} \supseteq \tilde{A}'_{n+1} \) and an extension of the
diagram by a map:

\[
\begin{array}{ccc}
C & \rightarrow & \bar{A}_{n+1}' \\
\downarrow & & \downarrow \\
A & \rightarrow & A_{n+1}'
\end{array}
\]

Next, we attach \((n + 2)\)-cells to \(A_{n+1}'\) to kill the kernel of \(\pi_{n+1}(A_{n+1}') \rightarrow \pi_{n+1}(A)\), giving an extension \(\bar{A}_{n+2}'\) together with a map to \(A\). Continuing like this, we obtain a sequence

\[
\begin{array}{ccc}
C & \rightarrow & A_{n+1}' \\
\downarrow & & \downarrow \\
A & \rightarrow & A_{n+1}'
\end{array}
\]

\[
\begin{array}{ccc}
C & \rightarrow & A_{n+2}' \\
\downarrow & & \downarrow \\
A & \rightarrow & A_{n+2}'
\end{array}
\]

and we let \(A'\) be the union (colimit) of this sequence with the weak topology. This space \(A'\) together with the induced maps \(C \rightarrow A' \rightarrow A\) verifies the assertion in the lemma.

If in Lemma 13.5 we start with a relative CW complex \(C \rightarrow A\), then by the relative version of the Whitehead Theorem, there exists a map \(e' : A \rightarrow A'\) with \(e' \circ i = i'\) and homotopies relative to \(C\) between \(e'e\) and \(1_A\), and between \(e'i\) and \(1_{A'}\). Thus, if we apply Lemma 13.5 to both \(C \rightarrow A\) and \(C \rightarrow B\) as in the statement of the excision theorem, we get homotopy equivalences \(e, e'\) and \(f, f'\) relative to \(C\),

\[
\begin{array}{ccc}
C & \rightarrow & B' \\
\downarrow & & \downarrow \\
A & \rightarrow & X \\
\downarrow & & \downarrow \\
A' & \rightarrow & X'
\end{array}
\]

and hence for \(X' = A' \cup_C B'\) well-defined homotopy equivalences

\[
\begin{array}{ccc}
X & \overset{e' \cup f'}{\sim} & X' \\
\sim & & \sim \\
X & \overset{e \cup f}{\sim} & X'
\end{array}
\]

Thus we conclude:

**Reduction 1.** It suffices to prove the excision theorem for extensions \(C \rightarrow A\) by cells of dimension larger than \(m\) and \(C \rightarrow B\) by cells of dimension larger than \(n\).

The next reduction concerns the number of cells one attaches to \(C\) to obtain \(A\) and \(B\) respectively. Let us say that a pair of extensions \(C \rightarrow A\) and \(C \rightarrow B\) as in Reduction 1 is of size \((p,q)\) if \(A\) is obtained by attaching \(p\) cells (of dimension \(> m\)) to \(C\), and \(B\) by attaching \(q\) cells (of dimension \(> n\)).
**Reduction 2.** If the excision theorem holds for extensions of size \((1,1)\), then it holds for extensions of arbitrary size \((p,q)\) with \(p,q \geq 1\).

**Proof.** Let us show by induction that the excision theorem holds for all extensions of type \((p,1)\). Given such an extension with \(p > 1\) let us write

\[
A = A' \cup e, \quad X' = A' \cup_C B
\]

so that we have two pushout squares

\[
\begin{array}{ccc}
C & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & X' \\
\downarrow & & \downarrow \\
A & \rightarrow & X
\end{array}
\]

in which the upper square is an extension of size \((p-1,1)\) and the lower one an extension of size \((1,1)\). By induction assumption the two maps of pairs \((A',C) \rightarrow (X',B)\) and \((A,A') \rightarrow (X,X')\) are \((m+n)\)-equivalences and we want to conclude the same for \((A,C) \rightarrow (X,B)\). The above diagram gives us a map of triples \((A,A',C) \rightarrow (X,X',B)\) so that we obtain by Lemma 13.6 the following diagram

\[
\begin{array}{ccccccc}
\pi_{i+1}(A,A') & \xrightarrow{\partial} & \pi_i(A',C) & \xrightarrow{i_*} & \pi_i(A,C) & \xrightarrow{j_*} & \pi_i(A,A') & \xrightarrow{\partial} & \pi_{i-1}(A',C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{i+1}(X,X') & \xrightarrow{\partial} & \pi_i(X',B) & \xrightarrow{w_*} & \pi_i(X,B) & \xrightarrow{v_*} & \pi_i(X,X') & \xrightarrow{\partial} & \pi_{i-1}(X',B).
\end{array}
\]

(Note that we needed the naturality statement of that lemma to get this commutative ladder.) To show that \(\gamma\) is an isomorphism in dimensions \(i < n + m\) it suffices to observe that our induction assumption guarantees that \(\beta, \delta\) and \(\epsilon\) are isomorphisms while \(\alpha\) is surjective. Thus, by the 5-Lemma (Lemma 13.7), we conclude that \(\gamma\) is an isomorphisms. Similarly, for \(i = m + n\) our induction assumption implies that \(\beta\) and \(\delta\) are surjective and that \(\epsilon\) is an isomorphism. Thus, again by the 5-Lemma, also the map \(\gamma\) is surjective and the map \((A,C) \rightarrow (X,B)\) is hence an \((m+n)\)-equivalence.

By a second induction over \(q\) we can now show that the excision theorem holds for all extensions of size \((p,q)\). In fact, if \(q > 1\) then let us write

\[
A = B' \cup e, \quad X'' = A \cup_C B'
\]

so that we have two pushout squares

\[
\begin{array}{ccc}
C & \rightarrow & B' \\
\downarrow & & \downarrow \\
A & \rightarrow & X''
\end{array} \quad \begin{array}{ccc} \rightarrow & B \\
\downarrow & \downarrow \\
X & \rightarrow
\end{array}
\]

The map \((A,C) \rightarrow (X,B)\) factors as \((A,C) \rightarrow (X'',B') \rightarrow (X,B)\). By induction assumption, both maps are \((m+n)\)-equivalences so that the same is the case for \((A,C) \rightarrow (X,B)\). 

□
Lemma 13.6. For pointed inclusions of subspaces $A \subseteq B \subseteq X$ the sequence of inclusions of pairs

$$(B, A) \xrightarrow{k} (X, A) \xrightarrow{l} (X, B)$$

induces a long exact sequence

$$\ldots \to \pi_i(B, A) \xrightarrow{k_*} \pi_i(X, A) \xrightarrow{l_*} \pi_i(X, B) \xrightarrow{\partial} \pi_{i-1}(B, A) \xrightarrow{k_*} \ldots$$

with $\partial$ being defined by $\pi_i(X, B) \xrightarrow{\partial} \pi_{i-1}(B) \to \pi_{i-1}(B, A)$. The connecting homomorphism is natural in maps of triples in the obvious sense.

Proof. We will just prove exactness at $\pi_i(X, A)$, and leave the other cases and the naturality as exercise. Also, we will write the proof for abelian groups, and leave as an exercise that the lemma also holds in those low degrees where one just has groups or even pointed sets (exercises!). It is clear that the composition

$$\pi_i(B, A) \xrightarrow{k_*} \pi_i(X, A) \xrightarrow{l_*} \pi_i(X, B)$$

is the zero map. To prove that $\ker(l_*) \subseteq \operatorname{im}(k_*)$, expand the diagram to

$$
\begin{array}{ccc}
\pi_i(B) & \xrightarrow{i_*} & \pi_i(X) & \xrightarrow{u_*} & \pi_i(X, B) \\
\downarrow{w_*} & & \downarrow{l_*} & & \downarrow{\partial} \\
\pi_i(B, A) & \xrightarrow{k_*} & \pi_i(X, A) & \xrightarrow{l_*} & \pi_i(X, B) \\
\downarrow{d} & & \downarrow{\partial} & & \downarrow{\partial} \\
\pi_{i-1}(A) & = & \pi_{i-1}(A) & = & \pi_{i-1}(B) \\
\end{array}
$$

where $\partial, \partial$, and $d$ are boundary maps for pairs, while we use the following names for inclusions:

$$
\begin{array}{ccc}
A & = & A \\
\downarrow{j} & & \downarrow{j} \\
B & = & B \\
\downarrow{w} & & \downarrow{w} \\
(X, A) & \xrightarrow{k} & (X, A) \\
\downarrow{t} & & \downarrow{t} \\
(X, B) & = & (X, B)
\end{array}
$$

Now suppose $x \in \pi_i(X, A)$ with $l_*(x) = 0$. Then $j_*\partial x = \partial l_* x = 0$, so $\partial x = d(y)$ for some $y \in \pi_i(B, A)$. Then $\partial(k_*(y) - x) = dy - \partial x = 0$, so $k_*(y) - x = u_*(z)$ for some $z \in \pi_i(X)$. But then $v_* z = l_* u_* z = l_* k_* y - l_* x = 0$, so $z = i_* t$ for some $t \in \pi_i(B)$. Now

$$x = k_* y - u_*(z) = k_* y - u_* i_* t = k_* (y - w_* t),$$

showing that any $x \in \ker(l_*)$ lies in the image of $k_*$. \qed

We refer to the sequence of Lemma 13.6 as the **long exact sequence of the (pointed) triple** $(X, B, A)$. Note that the definition of the connecting homomorphism of this sequence uses the connecting homomorphism of the long exact sequence of the pair $(X, B)$. 

**Lemma 13.7.** *(5-lemma)*

Consider a diagram of abelian groups (or groups) and homomorphisms with exact rows

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\
\alpha & \downarrow & \beta & \downarrow & \gamma & \downarrow & \delta & \downarrow & \epsilon \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E'.
\end{array}
\]

1. If \( \beta \) and \( \delta \) are surjective while \( \epsilon \) is injective then \( \gamma \) is surjective.
2. If \( \beta \) and \( \delta \) are injective while \( \alpha \) is surjective then \( \gamma \) is injective.

In particular, \( \beta \) and \( \delta \) are isomorphisms, \( \epsilon \) is injective, and \( \alpha \) is surjective, then also \( \gamma \) is an isomorphism.

**Proof.** Again, we write the proof for abelian groups and leave the verification for the relevant cases of groups and pointed sets as an exercise.

1. To check that \( \gamma \) is onto consider \( c' \in C' \). Since \( \delta \) is onto, there is a \( d \in D \) such that \( \delta(d) = h'(c') \). Then \( e(i(d)) = i'(\delta(d)) = i'(h'(c')) = 0 \). Since \( e \) is injective it follows that \( i(d) = 0 \), i.e., that \( d \in \ker(i) = \text{im}(h) \). Thus, there is a \( c \in C \) such that \( h(c) = d \). Now,

\[
h'(c' - \gamma(c)) = h'(c') - \delta(h(c)) = h'(c') - h'(c) = \delta(d) = h'(c) - h'(c') = 0,
\]

hence \( c' - \gamma(c) = g'(b') \) for some \( b' \in B' \). Since \( \beta \) is surjective there exists \( b \in B \) such that \( \beta(b) = b' \). We conclude that \( c' \in \text{im}(\gamma) \) by the final calculation

\[
\gamma(c + g(b)) = \gamma(c) + g'(\beta(b)) = \gamma(c) + g'(b') = \gamma(c) + (c' - \gamma(c)) = c'.
\]

2. Let \( c \in C \) be such that \( \gamma(c) = 0 \). Then \( h'(\gamma(c)) = \delta(h(c)) = 0 \). Since \( \delta \) is injective, we deduce that \( h(c) = 0 \) and hence \( c = g(b) \) for some \( b \in B \). It suffices to show that \( b \in \text{im}(f) \) since then \( g f = 0 \). Now \( g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0 \) tells us that \( \beta(b) = f'(a') \) for some \( a' \in A' \). Using the surjectivity of \( \alpha \) we conclude that \( a' = \alpha(a) \) for some \( a \in A \). Thus, the element \( f(a) \) satisfies \( \beta(f(a)) = f'(\alpha(a)) = \beta(b) \) and the injectivity of \( \beta \) implies \( f(a) = b \) as intended.

Combining these two statements immediately implies the remaining claim. \( \square \)

As a further special case, if the four morphisms \( \alpha, \beta, \delta, \) and \( \epsilon \) are isomorphisms then so is the fifth \( \gamma \).

After all these preparations, the proof of the Excision Theorem will be completed by the following proposition.

**Proposition 13.8.** Let \( C \) be a space, and define spaces \( A = C \cup e, B = C \cup e' \), and \( X = A \cup B \) by attaching cells of dimension \( > m \) and \( > m' \), respectively. Then \( \pi_i(A, C) \to \pi_i(X, B) \) is an isomorphism for \( i < n + m \) and a surjection for \( i \leq n + m \).

**Proof.** We will only prove surjectivity for \( i \leq n + m \). The proof of injectivity of \( i < n + m \) proceeds in exactly the same way, and is left as an exercise. If \( x \in e^o \) and \( y \in e'^o \) are points in the interior of the cells \( e \) and \( e' \), there is a diagram

\[
\begin{array}{ccc}
\pi_i(A, C) & \xrightarrow{\cong} & \pi_i(X, B) \\
\downarrow \cong & & \downarrow \cong \\
\pi_i(X - \{y\}, X - \{x, y\}) & \xrightarrow{\cong} & \pi_i(X, X - \{x\})
\end{array}
\]
where the vertical maps are isomorphisms. Indeed, the space $X - \{x\}$ is homotopy equivalent to $B$ because one can contract $e - \{x\}$ to its boundary; and similarly for $X - \{y\}$ and $X - \{x, y\}$. Consider an arbitrary map $f: I^i \to X$ which represents an element of $\pi_i(X, B)$ for $i \leq m + n$. This means that $f$ maps the top face $I^{i-1} \times \{1\}$ into $B$ and sends the rest of the boundary $J^{i-1} = I^{i-1} \times \{0\} \cup \partial(I^{i-1}) \times I$ to the base point $x_0$. By the above diagram, it suffices to prove that $f$ is homotopic to a map $h = h_1$ through a homotopy $h_s$, $s \in [0, 1]$ such that

(a) $h$ avoids the point $y$, i.e., $h: I^i \to X - \{y\}$,
(b) in addition, for every $s \in [0, 1]$, the restriction of $h_s$ to the top face of $I^i$ avoids the point $x$, and
(c) for each $s \in [0, 1]$, $h_s$ maps $J^{i-1}$ to the base point.

Let $e_{1/2}$ and $e_{1/2}'$ be small balls of radius $1/2$ (or for that matter, any non-empty open subsets whose closures are contained in the interior of $e$ and $e'$, respectively), and let

$$U = f^{-1}(e_{1/2}^o \cup e_{1/2}'^o).$$

We will now use the basic fact that any continuous map between manifolds can be approximated by a homotopic smooth map (see e.g. the book by Bott, Tu). Since $\tilde{U}$ is disjoint from $J^{i-1}$, this gives a map $g: I^i \to X$ such that $g = f$ on $J^{i-1}$, and the restriction of $g$ is smooth as a map $g: U \to e_{1/2}^o \cup e_{1/2}'^o$. Moreover, by choosing $g$ as well as the homotopy close to $f$, we can assume that the restriction of $g$ as well as that of the homotopy $g \simeq f$ to the top face of $I^i$ both avoid $x$. Let us write $V = g^{-1}(e_{1/4}^o)$ and $V' = g^{-1}(e_{1/4}')$. Then, since $g$ is close to $f$ we can assume that the closure of $V \cup V'$ is contained in $U$. In other words, $g$ is smooth over the entire preimage of these two small balls of diameter $1/4$. Also, write $\pi: I^i = I^{i-1} \times I \to I^{i-1}$ for the projection away from the last coordinate. We claim that there exist points $x$ and $y$ with

$$x \in e_{1/4}^o, \quad y \in e_{1/4}'^o, \quad \text{and} \quad \pi g(x) \cap \pi g(y) = \emptyset. \quad (1)$$

Indeed, let $V \times_{I^{i-1}} V'$ be the pullback along $\pi$, consisting of pairs $(v, v')$ with $v \in V, v' \in V'$ and $\pi(v) = \pi(v')$, and consider the smooth map

$$g \times g: V \times_{I^{i-1}} V' \to e_{1/4}^o \cup e_{1/4}'^o. \quad (2)$$

Then a pair of points $(x, y)$ satisfies (1) if and only if it is not in the image of the map (2). So we only need to check that $g \times g$ is not surjective. This is simply a matter of counting dimensions: $V \times_{I^{i-1}} V'$ is a manifold of dimension $i + 1$, and since $i \leq m + n$ we have

$$i + 1 < (m + 1) + (n + 1) = \dim(e) + \dim(e') = \dim(e_{1/4}^o \times e_{1/4}'^o).$$

So any regular value $(x, y)$ of (2) cannot be in its image (cf. Guillemin-Pollak, page 21). Since $\pi g^{-1}(x)$ and $\pi g^{-1}(y)$ are disjoint closed subsets of $I^{i-1}$, there exists a continuous (even smooth) map $\theta: I^{i-1} \to I$ with

$$\theta |_{\pi g^{-1}(x)} = 0 \quad \text{and} \quad \theta |_{\pi g^{-1}(y)} = 1. \quad \text{Now define the homotopy}$$

$$h_s: I^{i-1} \times I \to X, \quad s \in [0, 1],$$

by

$$h_s(z, t) = g(z, t - s \theta(z)).$$

Notice that $h_0 = g$. We claim that $h_s$ and $h_1 = h$ satisfy requirements (a)-(c).

For (a), suppose to the contrary that $h_1(z, t) = y$, i.e., $g(z, 1 - \theta(z)) = y$. Then $z \in \pi g^{-1}(y)$ so $\theta(z) = 1$, whence $g(z, 0) = y$. But $g$ (like $f$) maps the bottom face of $I^i = I^{i-1} \times I$ to the base point $x_0$ which does not lie in $e_{1/2}^o$, so this is impossible.
For (b) we argue similarly: Suppose $h_s(z, 1) = x$. Then $z \in \pi g^{-1}(x)$ so $\theta(z) = 0$, whence $g(z, 1 - s\theta(z)) = g(z, 1) = x$. This contradicts that $g$ avoids $x$ on the top face of $I^i$.

Finally, for (c), take $(z, t) \in J^{i-1}$. Then either $t = 0$ whence $h_s(z, t) = g(z, t)$, or $z \in \partial I^{i-1}$, and in both cases $h_s(z, t) = x_0$ because $g$ agrees with $f$ on $J^{i-1}$. □