Dynamical effects of stratification and rotation:

**Thermal wind balance and atmospheric circulation**

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Content of this lecture

- General circulation of the atmosphere  
- Hydrostatic and geostrophic balance  
- Pressure coordinate  
- Thermal wind balance and jets  
- Thermal wind balance in a cyclone  
- **Exercise**: testing the thermal wind balance equation using radiosonde observations
The general circulation of the atmosphere

The longitudinal mean value of \( u \)

Subtropical jets
Hydrostatic balance:

$$\rho g = -\frac{\partial p}{\partial z}$$

Geostrophic balance:

$$v_g = \frac{1}{f \rho} \left( \frac{\partial p}{\partial x} \right)_z; u_g = -\frac{1}{f \rho} \left( \frac{\partial p}{\partial y} \right)_z.$$
Pressure coordinate

\[ (\delta A)_{1-3} = (\delta A)_{1-2} + (\delta A)_{2-3} \]
\[ = (\delta A)_{y,z} + (\delta A)_{y,x} \]
\[ = \left( \frac{\delta A}{\delta x} \right)_{y,z} \delta x + \left( \frac{\delta A}{\delta z} \right)_{y,x} \delta z \]

Let \( A = p \):

\[ 0 = \left( \frac{\delta p}{\delta x} \right)_{y,z} + \left( \frac{\delta p}{\delta z} \right)_{y,x} \left( \frac{\delta z}{\delta x} \right)_{y,p} \]

Assuming hydrostatic equilibrium, \( \delta p/\delta z = -\rho g \):

\[ \left( \frac{\delta p}{\delta x} \right)_{y,z} = \rho g \left( \frac{\delta z}{\delta x} \right)_{y,p} \]

Pressure gradient term:

\[ \frac{\delta p}{\delta x} \]

In short:

\[ \frac{\delta p}{\delta x} = \rho g \left( \frac{\delta z}{\delta x} \right)_p \]
Geostrophic Balance

\[
\left( \frac{\partial p}{\partial x} \right)_z = \rho g \left( \frac{\partial z}{\partial x} \right)_p \quad \text{and} \quad \left( \frac{\partial p}{\partial y} \right)_z = \rho g \left( \frac{\partial z}{\partial y} \right)_p
\]

Geostrophic wind:
\[
v_g = \frac{1}{f \rho} \left( \frac{\partial p}{\partial x} \right)_z; \quad u_g = -\frac{1}{f \rho} \left( \frac{\partial p}{\partial y} \right)_z
\]

With pressure as a vertical coordinate:
\[
v_g = \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_p; \quad u_g = -\frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_p
\]

Geostrophic & Hydrostatic Balance in pressure coordinates

Geostrophic wind:
\[
v_g = \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_p; \quad u_g = -\frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_p
\]

Hydrostatic balance:
\[
\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad \frac{\partial z}{\partial \ln p} = -\frac{RT}{\rho g}
\]

Thermal wind equation:
\[
\frac{\partial v_g}{\partial \ln p} = -\frac{R \partial T}{f \partial x} \quad \text{and} \quad \frac{\partial u_g}{\partial \ln p} = \frac{R \partial T}{f \partial y}
\]
Thermal wind balance

Warm air: \[ \frac{\partial z}{\partial \ln p} = - \frac{RT}{g} \]

Cold air: \[ \frac{\partial z}{\partial \ln p} \text{ small} \]

\[ p_2 \]
\[ p_1 \]
\[ p_0 \]

NP \hspace{1cm} EQ

\[ u_s = -g \left( \frac{\partial z}{\partial y} \right)_f \]

\[ u_s \text{ increases with height if } f > 0 \]
Thermal wind balance

Isentropes (thin solid lines, labelled in Kelvin) and isotachs (isopleths of the velocity) (dashed lines, m s⁻¹) in a vertical section through a cold front. The y-coordinate is positive towards the left. Heavy lines mark the tropopause and frontal boundaries. The section extends approximately 1200 km in the horizontal direction (Palmen, E. and C.W. Newton, 1969: *Atmospheric Circulation Systems*. Academic Press, 603 pp).

Zonal mean temperature

ERA-40
Where do you expect the highest windspeeds?

Subtropical jets
Geopotential, $\Phi$

$$\Phi \equiv gz$$

$$v_g = \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_p$$
$$u_g = -\frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_p$$

$$f v_g = \left( \frac{\partial \Phi}{\partial x} \right)_p$$
$$fu_g = \left( \frac{\partial \Phi}{\partial y} \right)_p$$

Circumpolar flow at $p=250$ hPa

At which approximate height is $p=250$ hPa?
Geostrophic & Hydrostatic Balance in pressure coordinates

**Geostrophic wind:**

\[ v_g = \frac{g}{f} \left( \frac{\partial z}{\partial x} \right)_p \]
\[ u_g = -\frac{g}{f} \left( \frac{\partial z}{\partial y} \right)_p \]

**Hydrostatic balance:**

\[ \frac{\partial p}{\partial z} = -\rho g \]
\[ \frac{\partial z}{\partial \ln p} = -\frac{RT}{g} \]

**Thermal wind equation:**

\[ \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f} \frac{\partial T}{\partial x} \]
\[ \frac{\partial u_g}{\partial \ln p} = \frac{R}{f} \frac{\partial T}{\partial y} \]

Properties of the thermal wind

**Thermal wind equation:**

\[ \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f} \frac{\partial T}{\partial x} \]
\[ \frac{\partial u_g}{\partial \ln p} = \frac{R}{f} \frac{\partial T}{\partial y} \]

*Horizontal derivatives with pressure held constant

\[ \vec{k} \times \nabla \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \left( -\frac{\partial}{\partial y} \right)_x + \left( \frac{\partial}{\partial x} \right)_y \]

**In vector notation:**

\[ \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f} \vec{k} \times \vec{\nabla} T \]
Properties of the thermal wind

Thermal wind equation:

\[
\frac{\partial v_g}{\partial \ln p} = -\frac{R \partial T}{f \partial x} \quad \frac{\partial u_g}{\partial \ln p} = \frac{R \partial T}{f \partial y}
\]

*Horizontal derivatives with pressure held constant

In vector notation:

\[
\frac{\partial \vec{v}_g}{\partial \ln p} = -\frac{R}{f} \hat{k} \times \vec{\nabla}T
\]

The thermal wind:

\[
\vec{v}_T \equiv \vec{v}_g(p_1) - \vec{v}_g(p_0) = -\frac{R}{f} \int_{p_0}^{p_1} (\hat{k} \times \vec{\nabla}T) d\ln p
\]

The thermal wind is parallel to the isotherms!

Turning of the wind with height

Cold advection: backing with height

Warm advection: veering with height
The thermal wind in a mid-latitude cyclone

NOAA, channel 4 (infra-red). 12 Feb 1996, 1313 UTC

850 hPa
The thermal wind in a mid-latitude cyclone

Mid-latitude cyclone: conceptual model
Thermal wind in a mid-latitude cyclone

NOAA image in channel 4 (IR) made on March 3, 1995, at 0157 UTC.

Exercise 1:

Identify areas of warm air advection and areas of cold air advection between 850 hPa and 500 hPa over western Europe on 3 March 1995, 00 UTC (see following two slides).
Exercise 2:

Is the jet stream really in thermal wind balance?

Analysis of observations
Observations are performed at randomly spaced points (see upper panel of figure 1.77 and figure 1.78), while theoretical analysis and/or numerical models usually require data on a regular grid of points. This requires interpolation of the observations. The most simple interpolation method is called “piecewise linear interpolation”. Suppose that the gridpoint to which we want to interpolate the observations has the horizontal coordinates, \((x_0, y_0)\). The horizontal coordinates of three measuring points in the vicinity of this gridpoint are given by \((x_i, y_i)\), with \(i = 1, 2, 3\). The value of a variable (for instance the height of a pressure surface) at one of these points can be expressed as follows.

\[ z_i = z_0 + (x_i - x_0) \frac{\partial z}{\partial x} + (y_i - y_0) \frac{\partial z}{\partial y} \]

The desired value of the height at the gridpoint \((x_0, y_0)\) is \(z_0\). Applying this formula to the three measuring points yields three equations with three unknowns, \(z_0, \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}\), which can be solved, yielding estimates of values of \(z_0, \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}\), at the point \((x_0, y_0)\). An important action in this procedure, which precedes the actual interpolation, is the search of suitable measurements (i.e which three measurements are used for this interpolation?).

Compute the isobaric height \((z)\) at 700 hPa and at 500 hPa and wind \((u, v)\) at the point \((x_0, y_0) = (10.38^\circ E, 50.56^\circ N)\) (Meiningen) on 3 March 1995, 00 UTC. Also compute the isobaric height-gradients. We can use this information to compute the geostrophic wind. Compare the interpolated 500 hPa isobaric height with the measured height.

The observations can be obtained from from the following website:
http://weather.uwyo.edu/upperair/sounding.html.
Radiosonde observations of 3 March 1995, 00 UTC at 4 different stations including station code, latitude("N"), longitude ("E"). (at 500 and 400 hPa)

<table>
<thead>
<tr>
<th>PRES</th>
<th>HTGH</th>
<th>TEMP</th>
<th>DWPT</th>
<th>RELH</th>
<th>MIXR</th>
<th>DRCT</th>
<th>SKNT</th>
<th>THTA</th>
<th>THTE</th>
<th>THTV</th>
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<tbody>
<tr>
<td>hPa</td>
<td>m</td>
<td>°C</td>
<td>°C</td>
<td>%</td>
<td>g/kg</td>
<td>deg</td>
<td>knot</td>
<td>K</td>
<td>K</td>
<td>K</td>
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</table>

Station 0: 10548 Meiningen
Station latitude (\(f_0\)) = 50.56 Station longitude (\(l_0\)) = 10.38

<table>
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<tr>
<th>500.0</th>
<th>5370</th>
<th>-29.1</th>
<th>-34.0</th>
<th>63</th>
<th>0.43</th>
<th>260</th>
<th>99</th>
<th>297.5</th>
<th>298.0</th>
<th>297.6</th>
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<tbody>
<tr>
<td>400.0</td>
<td>6930</td>
<td>-41.9</td>
<td>-47.9</td>
<td>52</td>
<td>0.13</td>
<td>260</td>
<td>71</td>
<td>300.4</td>
<td>300.9</td>
<td>300.5</td>
</tr>
</tbody>
</table>

Station 1: 10739 Stuttgart
Station latitude (\(f_1\)) = 48.83 Station longitude (\(l_1\)) = 9.19

<table>
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<tr>
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<th>5440</th>
<th>-27.1</th>
<th>-32.1</th>
<th>62</th>
<th>0.52</th>
<th>299.9</th>
<th>301.7</th>
<th>300.0</th>
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<tbody>
<tr>
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<td>-49.9</td>
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<td>301.8</td>
<td>302.1</td>
<td>301.8</td>
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Station 2: 10393 Lindenberg
Station latitude (\(f_2\)) = 52.2 Station longitude (\(l_2\)) = 14.1

<table>
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<th>-31.1</th>
<th>-41.1</th>
<th>37</th>
<th>0.21</th>
<th>265</th>
<th>70</th>
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<td>6880</td>
<td>-41.5</td>
<td>-54.5</td>
<td>23</td>
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<td>71</td>
<td>301.0</td>
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Station 3: 10410 Essen (EDZE)
Station latitude (\(f_3\)) = 51.4 Station longitude (\(l_3\)) = 6.96

<table>
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<th>5300</th>
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<th>-33.3</th>
<th>71</th>
<th>0.46</th>
<th>245</th>
<th>52</th>
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<td>-47.4</td>
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<td>240</td>
<td>75</td>
<td>299.7</td>
<td>300.2</td>
<td>299.7</td>
</tr>
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</table>
Summary of the procedure and specific questions

The desired value of the height at the gridpoint \((x_0, y_0)\) or \((\lambda_0, \phi_0)\) is \(z_0\). Applying this formula to the three measuring points yields three equations with three unknowns: \(z_0\), \(\frac{\partial z}{\partial \lambda}\) and \(\frac{\partial z}{\partial \phi}\). These three equations can be written concisely as

\[
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} =
\begin{bmatrix}
1 & (\lambda_1 - \lambda_0) & (\phi_1 - \phi_0) \\
1 & (\lambda_2 - \lambda_0) & (\phi_2 - \phi_0) \\
1 & (\lambda_3 - \lambda_0) & (\phi_3 - \phi_0)
\end{bmatrix}
\begin{bmatrix}
z_0 \\
\frac{\partial z}{\partial \lambda} \\
\frac{\partial z}{\partial \phi}
\end{bmatrix}
\]

which can be written shortly as

\[
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} = M
\begin{bmatrix}
z_0 \\
\frac{\partial z}{\partial \lambda} \\
\frac{\partial z}{\partial \phi}
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
z_0 \\
\frac{\partial z}{\partial \lambda} \\
\frac{\partial z}{\partial \phi}
\end{bmatrix} = M^{-1}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix}
\]

We need to invert \(M\) (http://www.bluebit.gr/matrix-calculator/calculate.aspx) in order to calculate \(z_0\), \(\frac{\partial z}{\partial \lambda}\) and \(\frac{\partial z}{\partial \phi}\).

From \(\frac{\partial z}{\partial \lambda}\) and \(\frac{\partial z}{\partial \phi}\) we can obtain an estimate of the geostrophic wind for Meiningen from the geostrophic wind equation (\(a\) is the radius of the earth):

\[
\nu_g = \frac{g}{f} \frac{\partial z}{\partial x} \\
\nu_g = \frac{g}{fa \cos \phi} \frac{\partial z}{\partial \lambda} \\
\nu_g = -\frac{g}{f} \frac{\partial z}{\partial y} \\
\nu_g = -\frac{g}{fa \cos \phi} \frac{\partial z}{\partial \phi}
\]

If we apply this procedure to the 500 hPa and the 400 hPa pressure surfaces we can find the thermal wind between 500 and 400 hPa, \(\Delta \nu_g\) and \(\Delta \nu_g\).

According to the thermal wind equation,

\[
\Delta \nu_g = -\frac{R}{f} \frac{\partial T}{\partial \lambda} \Delta \ln p, \quad \Delta \mu_g = \frac{R}{f} \frac{\partial T}{\partial y} \Delta \ln p
\]

Next we apply the interpolation to the the average temperature in the layer between 500 and 400 hPa, which yields an estimate of the average temperature gradient. Use the thermal wind equations to obtain a new estimate of \(\Delta \mu_g\) and \(\Delta \nu_g\).

Compare these estimates with the measured values of \(\Delta \mu\) and \(\Delta \nu\).

Please be careful with the units and note that \(R=287\ J K^{-1} \ kg^{-1}\) and \(f=2\Omega \sin \phi\), with \(\Omega=0.00007292\ s^{-1}\) and \(\phi\) the latitude. The radius of the earth \(a=6370000\ m\) (at the latitude of Meiningen \(g/(fa)=0.0137\)).

Explain the differences between your theoretical result and the observations.
Analysis with “optimum linear interpolation” of the observed wind at 500 hPa

Analysis of 500 hPa isobaric surface height (contour interval is 25 m; thick contour corresponds to 5300 m), together with the geostrophic wind vector computed from the analysed isobaric height (with “optimum linear interpolation”).