Upward-driven disk: a mechanical model of convection

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Ocean Circulation

Meridional overturning (Thermo-Haline) circulation
Influenced by:
- Earth rotation
- Earth geometry (spherical shell)
- Ocean domains

Multiple steady states

Surface

Deep sea

Box model

Well mixed

Meridional Overturning
Gyres, eddies, waves

Drives by differential
- Heating
- Rain/evaporation/river run-off
- Momentum fluxes (wind)
- Tidal forces

Contains many scales. But rotation and stratification collect energy at largest scale!

Temperature T & salinity S determine density
\[ \rho = \rho_0 - a T + b S \]

Examples:
- Labrador Sea
- Greenland Sea
- Gulf of Lyon (Mediterranean)
- Antarctic (Weddell) Sea

Penetration depth varies yearly:
from <1200 m to >2300 m even without big atmospheric changes

Convection: efficient macroscopic heat transfer mechanism, driven by and advecting temperature field(s).

Ocean Engine

Opposite T and S effects on meridional density gradient

Convection: efficient macroscopic heat transfer mechanism, driven by and advecting temperature field(s).
**Rationale for restricted degrees-of-freedom model**: 
Stratification and (earth) rotation imply cascade of energy towards largest scale (basin scale)

Capture large-scale dynamics by using integral quantities to describe Ocean State and Circulation: ‘Moments’ $M_i$ of density and flow-fields

\[
\begin{align*}
M_m &= \int f(x) \, dx \quad & \text{Mass} \\
M_{(xi)} &= \int g(x) \, dx \quad & \text{Centre-of-mass (COM)} \\
M_{\sigma} &= \int \sigma \, f(x) \, dx \quad & \text{Variance}
\end{align*}
\]

**Stratified, single box model**
Navier-Stokes equations (PDEs) on $f$-plane forced by:
- differential buoyancy fluxes
- wind

+ salt & heat

Assume zeroth moments are constant (or zero)

Evolution equations (ODEs) for first moments of density and angular momentum fields:

\[
\begin{align*}
\dot{X} &= \frac{1}{\rho} \int x v(x,t) \, dV \\
\dot{L} &= \frac{1}{\rho} \int x \times \mathbf{v}(x,t) \, dV
\end{align*}
\]

\[
\begin{align*}
\dot{X} + X \times L &= -(X, Y, \mu Z) + Ra \mathbf{F} \\
Pr^{-1} \dot{L} + f \times L &= -(Y, X, 0) - (L_1, L_2, L_3) + TT
\end{align*}
\]

3D driven rotating pendulum → Multiple Equilibria

Maas 1994, 2004

**Non-rotating ($f=0$) convection**
- boiling of water in a pan
- heater in a room
- heating of the ground, leading to atmospheric vertical motions (often visualized by condensation in form of Cumulus clouds) – small scale
- ocean convection due to cooling & evaporation at ocean surface – large scale (rotation important)

**Stratified, single-box COM model**
NS + heat Eqs; here no rotation, only differential vertical buoyancy flux. Assume zeroth moments constant. Multiply by $\nu$ to derive ODEs for first moments of angular momentum and density fields:

buoyancy torque

\[
\begin{align*}
\frac{dX}{dt} &= Y - X, & \text{friction} \\
\frac{dY}{dt} &= XZ - Y, & \text{buoyancy forcing} \\
\frac{dZ}{dt} &= -XY - hZ + w, & \text{advection diffusion}
\end{align*}
\]

Maas 1994, 2004
Stratified, single-box COM model

NS + heat Eqns; here: no rotation, only differential vertical buoyancy flux. Assume zeroth moments constant. Multiply by \( \chi \) to derive ODEs for first moments of angular momentum and density fields:

\[
x(t) = \frac{\chi}{\rho_0} \int \rho(x,y,z) \, dx \, dy \, dz
\]

\[
y(t) = \frac{\chi}{\rho_0} \int \rho(x,y,z) \, dx \, dy \, dz
\]

\[
z(t) = \frac{\chi}{\rho_0} \int \rho(x,y,z) \, dx \, dy \, dz
\]

Maas 1994, 2004

Advection diffusion friction

Buoyancy torque

\[
\frac{dX}{dt} = Y - X
\]

\[
\frac{dY}{dt} = ZX - Y
\]

\[
\frac{dZ}{dt} = -XY - bZ + w
\]

Physical interpretation variables

Not 'upside down'

Explicit buoyancy forcing

Consider \((X,Y,Z)\) \(\rightarrow\) \(Pr(X,Y,Z)\);

\[
\lim_{Pr \to 0} \frac{w}{Pr}
\]

Van der Schrier & Maas 2000

Diffusionless Lorenz Equations: DLE

- Nonlinear problem with single-parameter: drive \( W \)

Looks so simple, we should be able to make this mechanically....

Upward-Driven Disk (UDD):
mechanical model of convection

- Constant upward drive
- Disc perpendicular to the traction wheel

Equations of motion

Moment of inertia: \( I = m \left( r^2 + d^2 \right) \)

\[
d = \sqrt{Y^2 + Z^2}
\]

\[
\frac{d(1\dot{\theta})}{dt} = mgY - k\dot{\theta}
\]

\[
\frac{dY}{dt} = Z\dot{\theta}
\]

\[
\frac{dZ}{dt} = -Y\dot{\theta} + W
\]

For \( d < -r \), \( I = mr^2 \) (constant)

Scaling --- DLE (with \( \dot{\theta} = X \))

Measurements

- RGB image analysis: compute centre-of-mass \( \Delta \)

Results

Drive at low speed \( \rightarrow \) aperiodic

- 1.4 cm/s
Drive at medium speed – irregular
- 3.9 cm/s

Drive at high speed – periodic
- 8.8 cm/s

DLE: numerical results

\[ W = \text{drive/friction} \]

\[
\begin{align*}
\frac{dX}{dt} &= Y - X \\
\frac{dY}{dt} &= ZX \\
\frac{dZ}{dt} &= -YX + W
\end{align*}
\]

Symmetric Periodic orbit \( W = \infty \)
Symmetry breaking bifurcations
Period-doubling bifurcations
Chaos \( W = O(1) \)

Selfsimilarity \( W \rightarrow 0 \)

Z-positions at \( Y=0 \)

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\text{Linearisation DLE, } W<<1

\[
\begin{align*}
\frac{dX}{dt} + \frac{dX}{dt} - W_0X &= 0 \\
\frac{d^2X}{dt^2} + \frac{dX}{dt} - W_0X &= 0
\end{align*}
\]

With \( X = e^{\gamma t} \zeta \) \rightarrow \( \frac{d^2\zeta}{dt^2} + \frac{1}{\gamma^2} + W_0 \zeta = 0 \) (Airy)

\[ \zeta = \alpha \text{Ai}(\frac{1}{4} + W) + \beta \text{Bi}(\frac{1}{4} + W) \]
Each cycle: determine $\alpha, \beta$ and period $T$

\[ X = e^{\alpha t} \left[ \alpha A \left( \frac{1}{4} + W_t \right) + \beta B \left( \frac{1}{4} - W_t \right) \right] \]

\[ Y = \frac{dX}{dt} + X \]

Energy equation

Requires matching (continuity) of $X$ and $dX/dt$ when $W=X^2$

Approx. solution DLE by iterating map

Measured and modelled

Unsuspected behaviour...

Conclusions

Stratification and rotation organize ocean circulation and convection in enclosed basins on largest scale, captured by dynamics centre-of-mass: in 2D, ‘Diffusionless Lorenz Equations’

Upward-driven disk models essence of 2D convection:
- upward drive centre-of-mass (differential heat flux)
- damping by mechanical friction

Features:
- steady (T), periodic and aperiodic response
- sensitive dependence on horizontal, off-axis displacement
- pre-existing horizontal density gradients: ‘preconditioning’
- Deep “flushes” when centre-of-mass starts close to symmetry axis

Future:
- Monitor disc’s displacement in bifurcation diagram for range of forcing strengths
- Compare to real convection experiments
- Add rotation to mechanical device

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