Derivation for the ratio of the sphere cap volume $V_{\text{cap}}$ to the sphere volume $V_{\text{sph}}$

The cosine of the wetting angle $\theta$ in Figure 2 equals:

$$\cos \theta = \frac{a - h}{a}; \quad h = a (1 - \cos \theta) \quad (A.1)$$

The volume of a sphere cap with height $h$ is:

$$V_{\text{cap}} = \frac{\pi}{3} h^2 (3a - h) = \frac{\pi}{3} a^3 \left(1 - \cos \theta \right)^2 \left(2 + \cos \theta \right). \quad (A.2)$$

where we have substituted $h$ from (A.1). Using the sphere volume $V_{\text{sph}} = (4/3)\pi a^3$, and working out the $\cos \theta$ term we find from (A.2):

$$\frac{V_{\text{cap}}}{V_{\text{sph}}} = \frac{1}{2} - \frac{1}{2} \cos \theta \left( 1 + \frac{1}{2} \sin^2 \theta \right) \quad (A.3)$$

The area of the sphere cap is:

$$A_{\text{cap}} = 2\pi ah = 2\pi a^2 \left(1 - \cos \theta \right) \quad (A.4)$$

The base area of the cap follows from Pythagoras' theorem:

$$A_{\text{base}} = \pi \left[ a^2 - (a - h)^2 \right] = \pi a^2 \left(1 - \cos^2 \theta \right) \quad (A.5)$$

From eqs. (A.4) and (A.5), together with the sphere area $A_{\text{sph}} = 4\pi a^2$, we find:

$$\frac{A_{\text{cap}} - A_{\text{base}} \cos \theta}{A_{\text{sph}}} = \frac{1}{2} - \frac{1}{2} \cos \theta \left( 1 + \frac{1}{2} \sin^2 \theta \right) \quad (A.6)$$

This is precisely the volume ratio in A.3, which proofs the identity:

$$\frac{V_{\text{cap}}}{V_{\text{sph}}} = \frac{A_{\text{cap}} - A_{\text{base}} \cos \theta}{A_{\text{sph}}} \quad (A.7)$$

that is used as eq 10 in the main text to prove eq 11.